Mass lumping and quadrature formulas

by Bogdan Radu

Model problem

1D wave equation:

$$\partial_{tt} u - \partial_{xx} u = 0$$
$$u(x,0) = u_0(x)$$
$$\partial_t u(x,0) = u_1(x)$$

in $\mathbb{R}$: $u(x,t) = u_0(x+t) + u_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\,ds$

on $[a,b]$: $u(x,t) = \ldots$ numerical methods instead!
or RHS

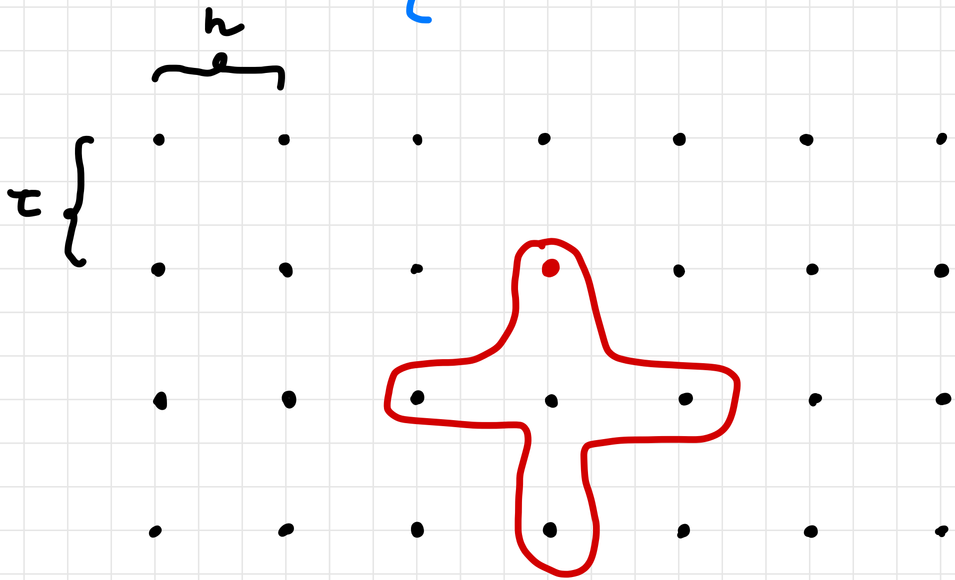
FD, FEM, ...

P1

Finite differences

$u_{tt} - u_{xx} = 0 \rightsquigarrow$

$$\frac{u_x^{n+1} - 2u_x^n + u_x^{n-1}}{\tau^2} = \frac{u_{x-1}^n - 2u_x^n + u_{x-1}^n}{h^2}$$



- higher order methods lead to larger stencils
- boundary conditions?

P2

Finite element method

Find $(u_h^n)_n \in V_h$

$$\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n+1}}{\tau^2}, v_h \right) + (\partial_x u_h^n, \partial_x v_h) = 0, \quad \forall v_h \in V_h$$

For a V_h and a set of basis functions, we get

mass matrix

stiffness matrix.

$$M \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + K u^n = 0$$

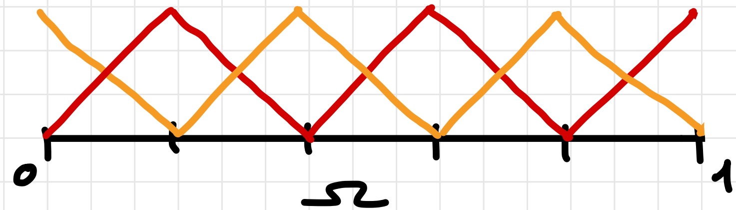
What do we know about M ?

Would be nice if it were easy to invert...

$$u^{n+1} = 2u^n - u^{n-1} + \tau^2 \boxed{M^{-1}} K u^n$$

Finite element method

Let $V_h = P_1(T_h) \cap H^1(\Omega)$



$$M = \begin{pmatrix} \times & \times & & & & \\ \times & \times & \times & & & \\ & \times & \times & \times & & \\ & & \times & \times & \times & \\ & & & \times & \times & \times \\ & & & & \times & \times \end{pmatrix}$$

$\int_\Omega \phi_6 \phi_5$

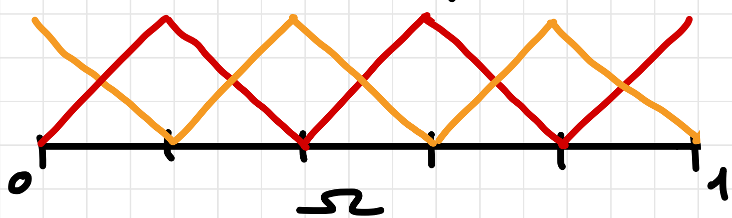
- Solving possible in $O(n)$, but not "optimal"

P4

Mass lumping : replace M by an approx M_h

- Idea : row-sum of M, i.e. $M_h := \mathrm{diag}(\mathrm{sum}(M))$
  - Pro : M_h is diagonal
  - Con : NO ANALYSIS

- Idea #2 : use numerical quadrature



$$(M_h)_{ij} = \sum_{T \in \mathcal{T}_h} \oint_T \phi_j \phi_i = \oint_{T^*} \phi_j \phi_i = \delta_{ij} \cdot |T^*| \cdot \frac{1}{2}$$

P5

Mass lumping :

$$M_h = \begin{pmatrix} \times & & & & & \\ & \times & & & & \\ & & \times & & & \\ & & & \times & & \\ & & & & \times & \\ & & & & & \times \end{pmatrix}$$

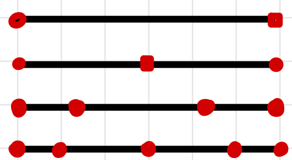
- Applying the inverse optimal, exactly n operations
- The additional error can be analyzed, no loss of convergence occurs.
- Surprisingly better CFL condition, allows for larger time steps.

What about higher order approximations ?

Higher order mass lumping in 1D

Let $V_h = \mathbb{P}_k(\mathcal{T}_h) \cap H^1(\Omega)$. A few remarks:

- For continuity reasons, we "must" include the vertices as quadrature points. ↪ Gauss-Lobatto



GL with $\boxed{k+1}$ points integrate all $\mathbb{P}_{2k-1}$ exactly

- $\dim \mathbb{P}_k = \boxed{k+1}$

- Lumped mass matrix M_h is diagonal, no loss of convergence due to mass lumping

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Wave equation 2D / 3D

$$\partial_{tt} u - \Delta u = 0$$
$$u(x,0) = u_0(x)$$
$$\partial_t u(x,0) = u_1(x)$$

- In $\mathbb{R}^2 / \mathbb{R}^3$: $u(x,t) = \dots$ "Kirchhoff's formulas"
- On bounded domains / inhomogeneous right-hand sides:

$u(x,t) = \dots$ numerical methods instead !

FD, FEM, ...

more flexible, allows for triangular grids, ...

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Wave equation 2D / 3D

$$\partial_{tt} u - \Delta u = 0$$

$$u(x,0) = u_0(x)$$

$$\partial_t u(x,0) = u_1(x)$$

- Variational formulation: Find $(u_h^n)_n \in V_h$

$$\left(\frac{u_h^{n-1} - 2u_h^n + u_h^{n+1}}{\tau^2}, v_h \right) + (\nabla u_h, \nabla v_h) = 0, \quad \forall v_h \in V_h$$

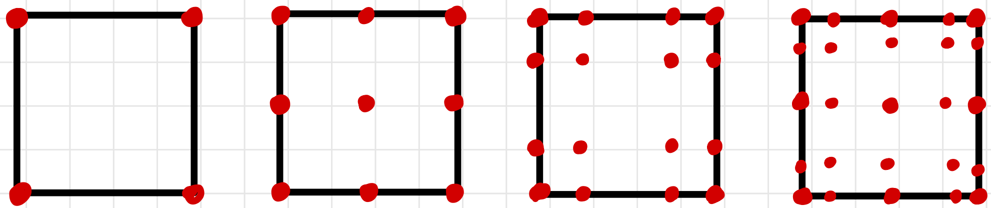
For a V_h and a set of basis functions, we get

mass matrix $\qquad\qquad\qquad\qquad$ stiffness matrix.

$$M \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + K u^n = 0$$

Mass lumping in 2D on quads

- We use tensor product Gauss-Lobatto quadrature

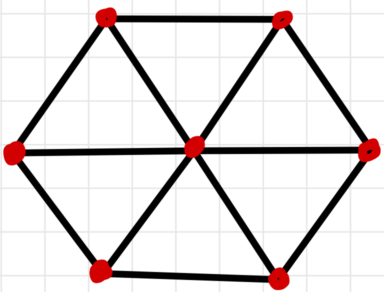


together with $V_h = Q_k(T_h) \cap H^1(\Omega)$

- Pro : can be again constructed for arbitrary orders
- But ... $\dim Q_k = (k+1)^2$, while $\dim P_k = \frac{1}{2}(k+1)(k+2)$
- What about triangles ?..

Mass lumping in 2D on triangles

- First order: Pick $V_h = P_1(T_h) \cap H^1(\Omega)$
- As quadrature rule, use the vertex rule



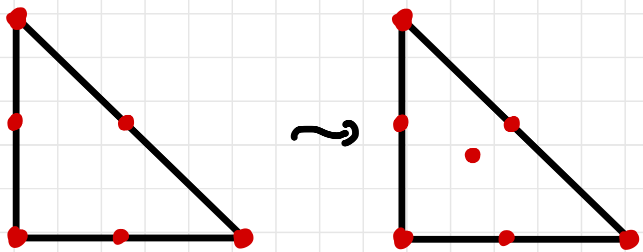
Local basis functions of the form

$\varphi_1(x,y) = x$, $\varphi_2(x,y) = y$, $\varphi_3(x,y) = 1 - x - y$

- Again, the lumped mass matrix M_h is diagonal
- What about higher orders? Gauss-Lobatto quadrature on triangles?

Higher order mass lumping on triangles

- Second order: Pick $V_h = P_2(T_h) \cap H^1(\Omega)$
- $\dim P_2 = 6$



$\rightsquigarrow$ but 7 quadrature points

only exact for P_1 | exact for P_3 polynomials

- Let $P_2^+(T) = P_2(T) + b_T$, $V_h^+ := P_2^+(T_h) \cap H^1(\Omega)$
- How do we do this in general?

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Higher order mass lumping on triangles

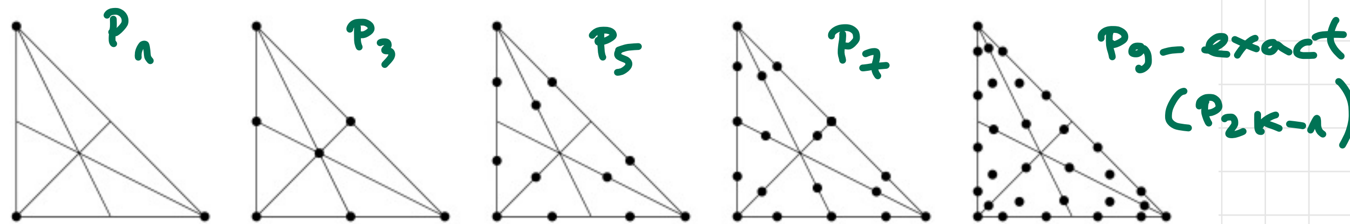
Conditions to be satisfied by the quadrature rule:

- vertices are quadrature points
- $k+1$ points on each edge (distributed symmetrically)
- positive weights (for stability of the method)
- integrates certain polynomials exactly

Higher order mass lumping on triangles

Short algorithm:

- Choose $V_h = B_k(T_h) \cap H^1(\Omega)$, $B_k(T_h) = P_k(T_h)$
- Find a quadrature rule as described above that integrates $k+k-2$ exactly and $\dim B_k = \#$ q.p.
- If not possible, increase $B_k(T) := B_k(T) + \text{bubbles}$, new k represents the highest polynomial order. Go to step 2.



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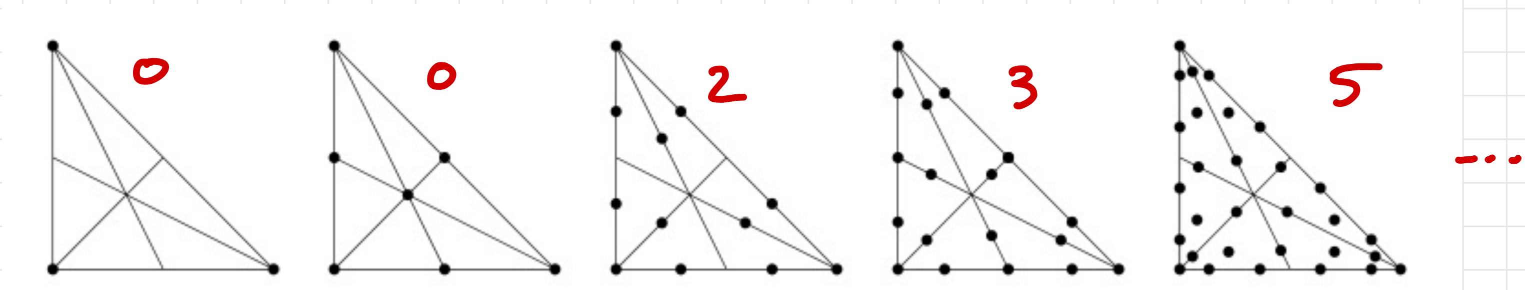
Higher order mass lumping on triangles

- No systematic way of determining these quadrature rules for arbitrary orders
- Increasingly more free parameters to vary



- For any given order, solve a non-linear problem to find weights and points ... Newton
- The 3D case ... just as complicated

P15

Maxwell's equations

$$\partial_{tt} E + \operatorname{curl} \operatorname{curl} E = 0 \quad \text{in } \Omega \times (0,t)$$

$$E(x,0) = E_0 \quad \text{in } \Omega$$

$$\partial_t E(x,0) = E_1 \quad \text{in } \Omega$$

- Variational formulation: Find $(u_h^n)_n \in V_h \subseteq H(\operatorname{curl}, \Omega)$

$$\left(\frac{u_h^{n-1} - 2u_h^n + u_h^{n+1}}{\tau^2}, v_h \right) + (\operatorname{curl} u_h, \operatorname{curl} v_h) = 0, \ \forall v_h \in V_h$$

mass matrix

stiffness matrix.

$$M \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + K u^n = 0$$

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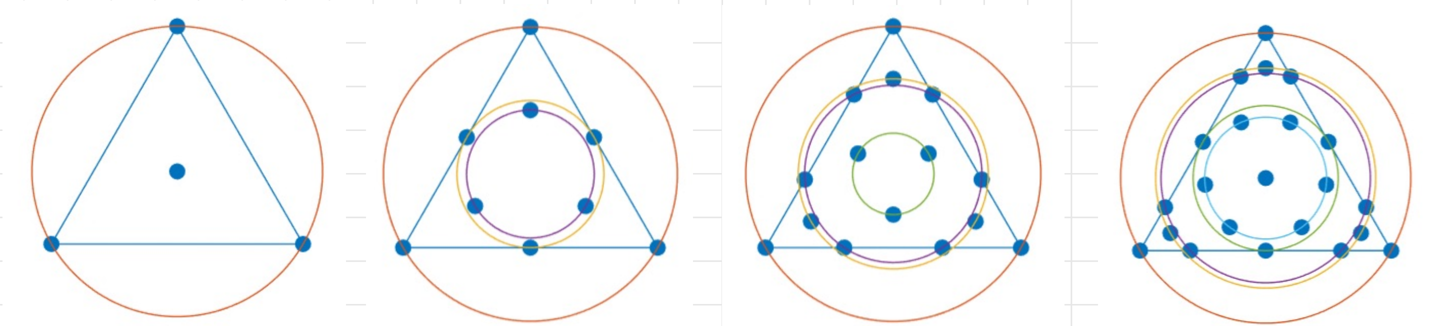
Mass lumping for Maxwell's equations

- Idea similar to H^1-conforming FEM, quadrature formulas different



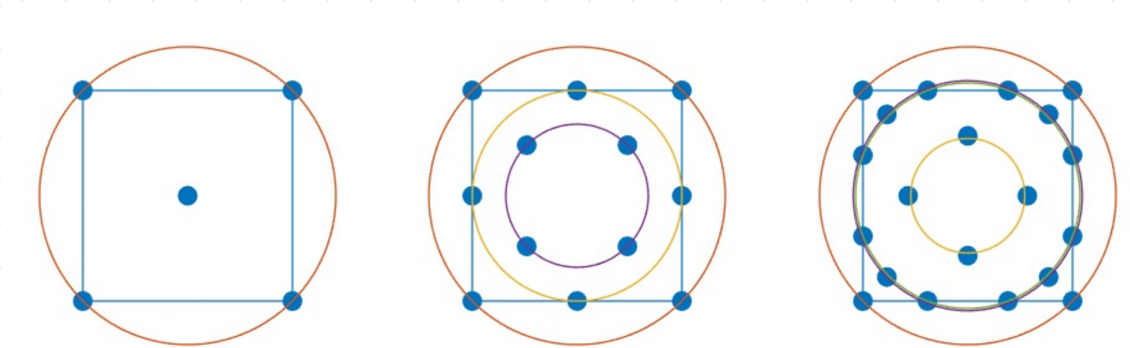
P_2 (0) P_4 (1) P_6 (3) P_8 - exact (6)

- different "sequences" of Gauss-Lobatto-type formulas

Mass lumping for Maxwell's equations

- different "sequences" of Gauss-Lobatto-type formulas

  ... even on quads/hexes



P_3 $\qquad$ P_5 $\qquad$ P_7 - exact

Takeaway

- Mass lumping is achieved by employing mass matrix approximations via quadrature rules
- No systematic way of designing arbitrary order Gauss-Lobatto quadrature formulas on simplices
- Looking towards a symbolic approach ... maybe?