A mixed finite element method with mass lumping for wave propagation

Herbert Egger Bogdan Radu

Graduate School of Computational Engineering
Technische Universität Darmstadt

21. November 2017

Acoustic wave equation

$$\begin{aligned} \partial_t u + \nabla p &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t p + \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \\ p &= 0 && \text{on } \partial\Omega \times (0, T). \end{aligned}$$

Acoustic wave equation

$$\begin{aligned} \partial_t u + \nabla p &= 0 && \text{in } \Omega \times (0,T), \\ \partial_t p + \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), \\ p &= 0 && \text{on } \partial\Omega \times (0,T). \end{aligned}$$

Remark (Existence and uniqueness)

Existence and uniqueness of a solution

$$(u,p) \in C([0,T], H(\operatorname{div},\Omega) \times H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)^2 \times L^2(\Omega))$$

for suitable initial and right hand side data follows from the semigroup theory.

Variational formulation

$$\begin{aligned} \partial_t u + \nabla p &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t p + \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \\ p &= 0 && \text{on } \partial\Omega \times (0, T). \end{aligned}$$

Variational characterization

$$\begin{aligned} (\partial_t u(t), v) - (p(t), \operatorname{div} v) &= 0 \quad \forall v \in H(\operatorname{div}, \Omega) \\ (\partial_t p(t), q) + (\operatorname{div} u(t), q) &= 0 \quad \forall q \in L^2(\Omega) \end{aligned}$$

Variational formulation

$$\begin{aligned} \partial_t u + \nabla p &= 0 && \text{in } \Omega \times (0, T), \\ \partial_t p + \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \\ p &= 0 && \text{on } \partial\Omega \times (0, T). \end{aligned}$$

Variational characterization

$$\begin{aligned} (\partial_t u(t), v) - (p(t), \operatorname{div} v) &= 0 \quad \forall v \in H(\operatorname{div}, \Omega) \\ (\partial_t p(t), q) + (\operatorname{div} u(t), q) &= 0 \quad \forall q \in L^2(\Omega) \end{aligned}$$

Remark

Each classical solution satisfies the variational characterization.

Remark

The spaces corresponding to the weak formulation are $L^2(\Omega)$ for p and $H(\operatorname{div}, \Omega)$ for u.

Discrete spaces

$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h) \qquad \mathrm{div}\, V_h = Q_h$$

Discrete spaces

$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h) \qquad \mathrm{div}\, V_h = Q_h$$

Projection operators $\rho_h : H^1(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \to V_h$ and $\pi_h^0 : L^2(\Omega) \to Q_h$

$$\mathrm{div}\, \rho_h v = \pi_h^0 \mathrm{div}\, v$$

$$\|u - \rho_h u\|_{L^2(\Omega)} \leq Ch^2 |u|_{2,\Omega},$$

$$\|p - \pi_h^0 p\|_{L^2(\Omega)} \leq Ch |p|_{1,\Omega}.$$

Discrete spaces

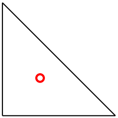
$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h) \qquad \mathrm{div}\; V_h = Q_h$$

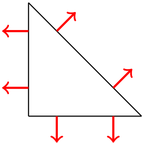
Projection operators $\rho_h : H^1(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \to V_h$ and $\pi_h^0 : L^2(\Omega) \to Q_h$

$$\mathrm{div}\; \rho_h v = \pi_h^0 \mathrm{div}\; v$$

$$\|u - \rho_h u\|_{L^2(\Omega)} \leq Ch^2 |u|_{2,\Omega},$$

$$\|p - \pi_h^0 p\|_{L^2(\Omega)} \leq Ch |p|_{1,\Omega}.$$





Discrete spaces

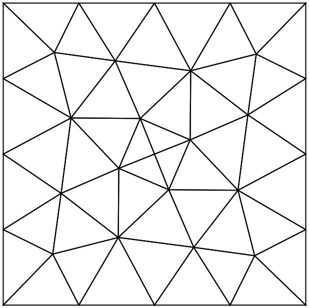
$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\mathrm{div}\,,\Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h) \qquad \mathrm{div}\, V_h = Q_h$$

Projection operators $\rho_h : H^1(\mathcal{T}_h) \cap H(\mathrm{div}\,,\Omega) \to V_h$ and $\pi_h^0 : L^2(\Omega) \to Q_h$

$$\mathrm{div}\, \rho_h v = \pi_h^0 \mathrm{div}\, v$$

$$\|u - \rho_h u\|_{L^2(\Omega)} \leq Ch^2 |u|_{2,\Omega},$$

$$\|p - \pi_h^0 p\|_{L^2(\Omega)} \leq Ch|p|_{1,\Omega}.$$



Discrete spaces

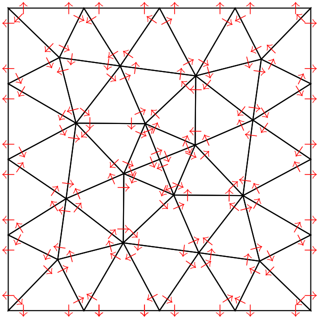
$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\mathrm{div}\,,\Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h) \qquad \mathrm{div}\, V_h = Q_h$$

Projection operators $\rho_h : H^1(\mathcal{T}_h) \cap H(\mathrm{div}\,,\Omega) \to V_h$ and $\pi_h^0 : L^2(\Omega) \to Q_h$

$$\mathrm{div}\,\rho_h v = \pi_h^0 \mathrm{div}\, v$$

$$\|u - \rho_h u\|_{L^2(\Omega)} \leq Ch^2 |u|_{2,\Omega},$$

$$\|p - \pi_h^0 p\|_{L^2(\Omega)} \leq Ch |p|_{1,\Omega}.$$



Semi-discretization

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Semi-discretization

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$\begin{aligned} (\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\ (\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h. \end{aligned}$$

Theorem (Error estimate for the semi-discretization)

Let $V_h = BDM_1$, $Q_h = P_0$. *Then if* (u, p) *sufficiently smooth*

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u, p)h.$$

Semi-discretization

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

Let $V_h = BDM_1$*,* $Q_h = P_0$*. Then if* (u, p) *sufficiently smooth*

$$
\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u,p)h^2
$$

Semi-discretization

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$\begin{aligned}(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\ (\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.\end{aligned}$$

Theorem (Error estimate for the semi-discretization)

Let $V_h = BDM_1$, $Q_h = P_0$. *Then if* (u, p) *sufficiently smooth*

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u, p) h^2$$

Theorem (Post-processing for the pressure)

For (u, p) *sufficiently smooth, we have*

$$\|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq C(p, u) h^2$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
M\dot{\mathbf{u}}_h + B\mathbf{p}_h &= 0 \\
D\dot{\mathbf{p}}_h - B^\top \mathbf{u}_h &= 0
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
M\frac{\mathbf{u}_h^{n+1/2}-\mathbf{u}_h^{n-1/2}}{\tau}+B\mathbf{p}_h^n &= 0\\
D\frac{\mathbf{p}_h^{n+1}-\mathbf{p}_h^{n}}{\tau}-B^\top\mathbf{u}_h^{n+1/2} &= 0
\end{aligned}
$$

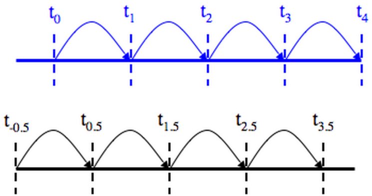
Compute $(u_h^{n+1/2}, p_h^{n+1})$ from $(u_h^{n-1/2}, p_h^{n})$.

Motivation for new method

$$M\frac{\mathbf{u}_h^{n+1/2}-\mathbf{u}_h^{n-1/2}}{\tau}+B\mathbf{p}_h^n = 0$$

$$D\frac{\mathbf{p}_h^{n+1}-\mathbf{p}_h^{n}}{\tau}-B^\top\mathbf{u}_h^{n+1/2} = 0$$

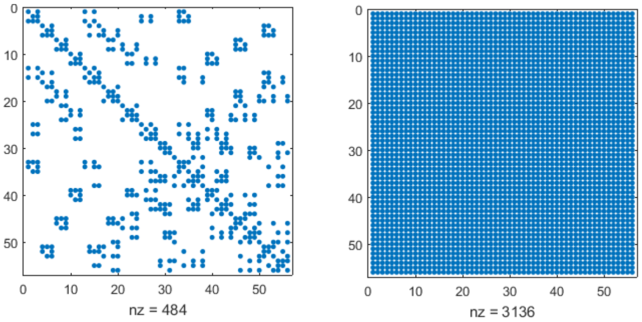
Compute $(u_h^{n+1/2}, p_h^{n+1})$ from $(u_h^{n-1/2}, p_h^n)$.



Motivation for new method

$$
\begin{aligned}
M\frac{\mathbf{u}_h^{n+1/2}-\mathbf{u}_h^{n-1/2}}{\tau} + B\mathbf{p}_h^n &= 0 \\
D\frac{\mathbf{p}_h^{n+1}-\mathbf{p}_h^{n}}{\tau} - B^\top \mathbf{u}_h^{n+1/2} &= 0
\end{aligned}
$$

Problem : Structure of matrix M



Motivation for new method

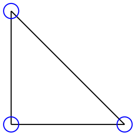
$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

$$
(u_h, v_h)_h := \frac{|T|}{3} \sum_{i=1}^{3} u_h(r_i) v_h(r_i)
$$

where r_i represent the edges of the element.

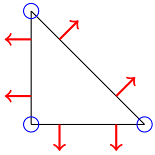


Motivation for new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

$$(u_h, v_h)_h := \frac{|T|}{3} \sum_{i=1}^{3} u_h(r_i) v_h(r_i)$$

where r_i represent the edges of the element.

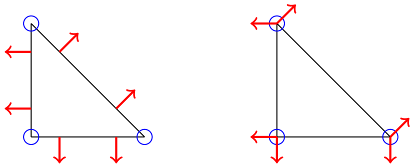


Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

$$
(u_h, v_h)_h := \frac{|T|}{3} \sum_{i=1}^{3} u_h(r_i) v_h(r_i)
$$

where r_i represent the edges of the element.

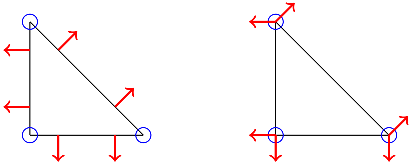


Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

$$
(u_h, v_h)_h := \frac{|T|}{3} \sum_{i=1}^{3} u_h(r_i) v_h(r_i)
$$

where r_i represent the edges of the element.

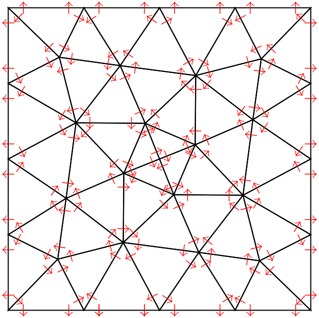


Norm equivalence

$$
\frac{1}{2} \|v_h\|_{h,\Omega} \leq \|v_h\|_{L^2(\Omega)} \leq \|v_h\|_{h,\Omega}, \qquad \forall v_h \in V_h.
$$

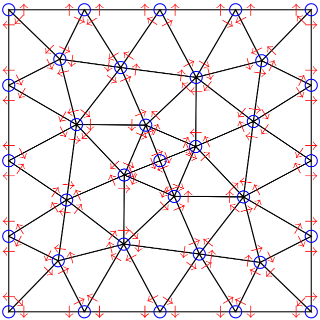
Motivation for new method

$$\color{red}{(\partial_t u_h(t), v_h)_h} - (p_h(t), \operatorname{div} v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$



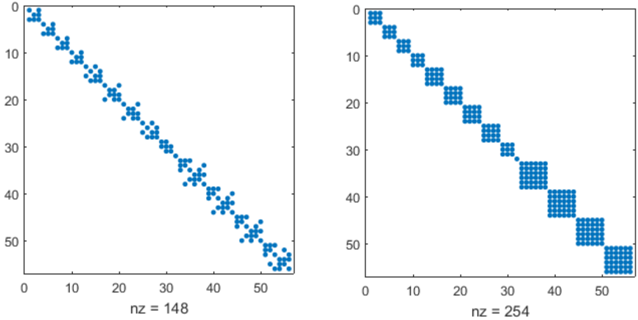
Motivation for new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$



Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$



Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

What do we lose ? Let's take a look at what the numerical tests have to say...

$$
\|\widetilde{\pi}_h^0 u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch \qquad \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq Ch^2.
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

What do we lose ? Let's take a look at what the numerical tests have to say...

$$
\|\widetilde{\pi}_h^0 u(t) - u_h(t)\|_{L^2(\Omega)} \leq Ch \qquad \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq Ch^2.
$$

WHY ?

Motivation for new method

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

Let $V_h = BDM_1$, $Q_h = P_0$. *Then if* (u, p) *sufficiently smooth*

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u, p)h.$$

Motivation for new method

$$\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 && \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 && \forall q_h \in Q_h.
\end{aligned}$$

Motivation for new method

$$(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Lemma (Discrete energy estimate)

Let (u_h, p_h) denote the solution of the system above. Then

$$\|u_h(t)\|_{L^2(\Omega)} + \|p_h(t)\|_{L^2(\Omega)}$$
$$\lesssim \|u_h(0)\|_{L^2(\Omega)} + \|p_h(0)\|_{L^2(\Omega)}$$

Proof idea : Test equations with $q_h = p_h(t)$ and $v_h = u_h(t)$.

Motivation for new method

$$\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) &= (f(t), v_h) \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) &= (g(t), q_h) \qquad \forall q_h \in Q_h.
\end{aligned}$$

Lemma (Discrete energy estimate)

Let (u_h, p_h) denote the solution of the system above. Then

$$\begin{aligned}
&\|u_h(t)\|_{L^2(\Omega)} + \|p_h(t)\|_{L^2(\Omega)}\\
&\lesssim \|u_h(0)\|_{L^2(\Omega)} + \|p_h(0)\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} + \|g(s)\|_{L^2(\Omega)} ds
\end{aligned}$$

Proof idea : Test equations with $q_h = p_h(t)$ and $v_h = u_h(t)$.

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= (f(t), v_h) \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= (g(t), q_h) \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Lemma (Discrete energy estimate)

Let (u_h, p_h) *denote the solution of the system above. Then*

$$
\begin{aligned}
&\|u_h(t)\|_{L^2(\Omega)} + \|p_h(t)\|_{L^2(\Omega)} \\
&\lesssim \|u_h(0)\|_{L^2(\Omega)} + \|p_h(0)\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} + \|g(s)\|_{L^2(\Omega)} ds
\end{aligned}
$$

Proof idea : Test equations with $q_h = p_h(t)$ and $v_h = u_h(t)$.

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\rho_h u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u,p)h^2
$$

For $w_h(t) = \rho_h u(t) - u_h(t)$ and $r_h(t) = \pi_h^0 p(t) - p_h(t)$, we have

$$
\begin{aligned}
(\partial_t w_h(t), v_h) - (r_h(t), \operatorname{div} v_h) &= (\widetilde{f}(t), v_h)\\
(\partial_t r_h(t), q_h) + (\operatorname{div} w_h(t), q_h) &= (\widetilde{g}(t), q_h)
\end{aligned}
$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$
\begin{aligned}
(\widetilde{f}(t), v_h) &= (\partial_t(\rho_h u(t) - u(t)), v_h) + (\pi_h^0 p(t) - p(t), \operatorname{div}\ v_h)\\
(\widetilde{g}(t), q_h) &= (\partial_t(\pi_h^0 p(t) - p(t)), q_h) + (\operatorname{div} \rho_h u(t) - \operatorname{div} u(t)), q_h) = 0
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\rho_h u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \le C(u,p)h^2
$$

For $w_h(t) = \rho_h u(t) - u_h(t)$ and $r_h(t) = \pi_h^0 p(t) - p_h(t)$, we have

$$
\begin{aligned}
(\partial_t w_h(t), v_h) - (r_h(t), \mathrm{div}\, v_h) &= (\widetilde{f}(t), v_h)\\
(\partial_t r_h(t), q_h) + (\mathrm{div}\, w_h(t), q_h) &= (\widetilde{g}(t), q_h)
\end{aligned}
$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$
\begin{aligned}
(\widetilde{f}(t), v_h) &\le Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)}\\
(\widetilde{g}(t), q_h) &= 0
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\rho_h u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq \dots
$$

For $w_h(t) = \rho_h u(t) - u_h(t)$ and $r_h(t) = \pi_h^0 p(t) - p_h(t)$, we have

$$
\begin{aligned}
(\partial_t w_h(t), v_h)_h - (r_h(t), \operatorname{div} v_h) &= (\widetilde{f}(t), v_h) \\
(\partial_t r_h(t), q_h) + (\operatorname{div} w_h(t), q_h) &= (\widetilde{g}(t), q_h)
\end{aligned}
$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$
\begin{aligned}
(\widetilde{f}(t), v_h) &\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + (\rho_h \partial_t u(t), v_h) - (\rho_h \partial_t u(t), v_h)_h \\
&\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + Ch \|\partial_t u(t)\|_{H^1(\Omega)} \|v_h\|_{L^2(\Omega)} \\
(\widetilde{g}(t), q_h) &= 0
\end{aligned}
$$

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\rho_h u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u,p)h
$$

For $w_h(t) = \rho_h u(t) - u_h(t)$ and $r_h(t) = \pi_h^0 p(t) - p_h(t)$, we have

$$
\begin{aligned}
(\partial_t w_h(t), v_h)_h - (r_h(t), \operatorname{div} v_h) &= (\widetilde{f}(t), v_h) \\
(\partial_t r_h(t), q_h) + (\operatorname{div} w_h(t), q_h) &= (\widetilde{g}(t), q_h)
\end{aligned}
$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$
\begin{aligned}
(\widetilde{f}(t), v_h) &\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + (\rho_h \partial_t u(t), v_h) - (\rho_h \partial_t u(t), v_h)_h \\
&\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + Ch \|\partial_t u(t)\|_{H^1(\Omega)} \|v_h\|_{L^2(\Omega)} \\
(\widetilde{g}(t), q_h) &= 0
\end{aligned}
$$

Taking a step back ...

Consider the elliptic projection

$$
\begin{aligned}
(w_h, v_h) - (r_h, \mathrm{div}\ v_h) &= (w, v_h) - (r, \mathrm{div}\ v_h) \qquad && \forall v_h \in V_h,\\
(\mathrm{div}\ w_h, q_h) &= (\mathrm{div}\ w, q_h) && \forall q_h \in Q_h.
\end{aligned}
$$

Taking a step back ...

Consider the elliptic projection

$$\begin{aligned} (w_h, v_h)_h - (r_h, \operatorname{div} v_h) &= (w, v_h) - (r, \operatorname{div} v_h) && \forall v_h \in V_h, \\ (\operatorname{div} w_h, q_h) &= (\operatorname{div} w, q_h) && \forall q_h \in Q_h. \end{aligned}$$

Taking a step back ...

Consider the elliptic projection

$$\begin{aligned} {\color{red}(w_h, v_h)_h} - (r_h, \mathrm{div}\ v_h) &= (w, v_h) - (r, \mathrm{div}\ v_h) \qquad && \forall v_h \in V_h, \\ (\mathrm{div}\ w_h, q_h) &= (\mathrm{div}\ w, q_h) && \forall q_h \in Q_h. \end{aligned}$$

Lemma (Estimates for the elliptic projection)

If Ω is convex, we have

$$\|\pi_h^0 r - r_h\|_{L^2(\Omega)} \le Ch^2 \left(\|w\|_{H^1(\Omega)} + \|\mathrm{div}\ w\|_{H^1(\Omega)} \right).$$

whenever w and r are sufficiently smooth.

Vague idea : Duality arguments and

$$|(u_h, v_h) - (u_h, v_h)_h| \le \begin{cases} Ch \|u_h\|_{H^1(\Omega)} \|v_h\|_{L^2(\Omega)} \\ C{\color{red}h^2} \|u_h\|_{H^1(\Omega)} \|v_h\|_{\color{red}H^1(\Omega)} \end{cases}$$

M. Wheeler and I. Yotov.
A multipoint flux mixed finite element method.
SIAM J. Numer. Anal., Vol. 44, No. 5, pp. 2082–2106.

Auxiliary functions

Consider the functions $u_h^* \in C^1(0, T; V_h)$ and $p_h^* \in C(0, T; Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \mathrm{div}\ v_h) &= (\partial_t u(t), v_h) - (p(t), \mathrm{div}\ v_h) \qquad && \forall v_h \in V_h \\
(\mathrm{div}\ \partial_t u_h^*(t), q_h) &= (\mathrm{div}\ \partial_t u(t), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

Auxiliary functions

Consider the functions $u_h^* \in C^1(0, T; V_h)$ and $p_h^* \in C(0, T; Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \operatorname{div} v_h) &= 0 && \forall v_h \in V_h \\
(\operatorname{div} \partial_t u_h^*(t), q_h) &= (\operatorname{div} \partial_t u(t), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

Auxiliary functions

Consider the functions $u_h^* \in C^1(0,T;V_h)$ and $p_h^* \in C(0,T;Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \mathrm{div}\ v_h) &= 0 && \forall v_h \in V_h \\
(\mathrm{div}\ \partial_t u_h^*(t), q_h) &= (\mathrm{div}\ \partial_t u(t), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

and

$$
\begin{aligned}
(u_h^*(0), v_h)_h - (r_h^*(0), \mathrm{div}\ v_h) &= (u(0), v_h) && \forall v_h \in V_h \\
(\mathrm{div}\ u_h^*(0), q_h) &= (\mathrm{div}\ u(0), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

Lemma (Approximation error estimates)

We have

$$
(\mathrm{div}\ u_h^*(t) - \mathrm{div}\ u(t), q_h) = 0, \quad \forall q_h \in Q_h
$$

Moreover, if Ω is convex, we have

$$
\|\pi_h^0 p(t) - p_h^*(t)\|_{L^2(\Omega)} \leq Ch^2 \left(\|\partial_t u(t)\|_{H^1(\Omega)} + \|\mathrm{div}\ \partial_t u(t)\|_{H^1(\Omega)}\right)
$$

whenever u is sufficiently smooth.

Motivation for new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\rho_h u(t) - u_h(t)\|_{L^2(\Omega)} + \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u,p)h
$$

For $w_h(t) = \rho_h u(t) - u_h(t)$ and $r_h(t) = \pi_h^0 p(t) - p_h(t)$, we have

$$
\begin{aligned}
(\partial_t w_h(t), v_h)_h - (r_h(t), \operatorname{div} v_h) &= (\widetilde{f}(t), v_h)\\
(\partial_t r_h(t), q_h) + (\operatorname{div} w_h(t), q_h) &= (\widetilde{g}(t), q_h)
\end{aligned}
$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$
\begin{aligned}
(\widetilde{f}(t), v_h) &\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + (\rho_h \partial_t u(t), v_h) - (\rho_h \partial_t u(t), v_h)_h\\
&\leq Ch^2 \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)} + Ch \|\partial_t u(t)\|_{H^2(\Omega)} \|v_h\|_{L^2(\Omega)}\\
(\widetilde{g}(t), q_h) &= 0
\end{aligned}
$$

A new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Theorem (Error estimate for the semi-discretization)

$$\|u_h^*(t) - u_h(t)\|_{L^2(\Omega)} + \|p_h^*(t) - p_h(t)\|_{L^2(\Omega)} \leq \dots$$

For $w_h(t) = u_h^*(t) - u_h(t)$ and $r_h(t) = p_h^*(t) - p_h(t)$, we have

$$(\partial_t w_h(t), v_h)_h - (r_h(t), \mathrm{div}\, v_h) = (\widetilde{f}(t), v_h)$$
$$(\partial_t r_h(t), q_h) + (\mathrm{div}\, w_h(t), q_h) = (\widetilde{g}(t), q_h)$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$(\widetilde{f}(t), v_h) = (\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \mathrm{div}\ v_h) = 0$$
$$(\widetilde{g}(t), q_h) = (\partial_t p_h^*(t) - \partial_t p(t), q_h) + (\mathrm{div}\, u_h^*(t) - \mathrm{div}\, u(t), q_h)$$
$$\leq \|\partial_t p_h^*(t) - \partial_t \pi_h^0 p(t)\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}$$

A new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Theorem (Error estimate for the semi-discretization)

$$\|u_h^*(t) - u_h(t)\|_{L^2(\Omega)} + \|p_h^*(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u)h^2$$

For $w_h(t) = u_h^*(t) - u_h(t)$ and $r_h(t) = p_h^*(t) - p_h(t)$, we have

$$(\partial_t w_h(t), v_h)_h - (r_h(t), \mathrm{div}\, v_h) = (\widetilde{f}(t), v_h)$$
$$(\partial_t r_h(t), q_h) + (\mathrm{div}\, w_h(t), q_h) = (\widetilde{g}(t), q_h)$$

with initial values $w_h(0) = 0$, $r_h(0) = 0$ and right hand sides

$$(\widetilde{f}(t), v_h) = (\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \mathrm{div}\ v_h) = 0$$
$$(\widetilde{g}(t), q_h) = (\partial_t p_h^*(t) - \partial_t p(t), q_h) + (\mathrm{div}\, u_h^*(t) - \mathrm{div}\, u(t), q_h)$$
$$\leq \|\partial_t p_h^*(t) - \partial_t \pi_h^0 p(t)\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}$$

A new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) &= 0 \qquad \forall v_h \in V_h, \\
(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u) h^2
$$

Proof: Split $\|\pi_h^0 p(t) - p_h(t)\| \leq \|\pi_h^0 p(t) - p_h^*(t)\| + \|p_h^*(t) - p_h(t)\|$.

A new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Theorem (Error estimate for the semi-discretization)

$$\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u)h^2$$

Proof: Split $\|\pi_h^0 p(t) - p_h(t)\| \leq \|\pi_h^0 p(t) - p_h^*(t)\| + \|p_h^*(t) - p_h(t)\|$.

What is $C(u)$? One of the terms it contains is

$$\|\mathrm{div}\ \partial_{tt} u(t)\|_{H^1(\Omega)}$$

A new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Theorem (Error estimate for the semi-discretization)

$$\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u)h^2$$

Proof: Split $\|\pi_h^0 p(t) - p_h(t)\| \leq \|\pi_h^0 p(t) - p_h^*(t)\| + \|p_h^*(t) - p_h(t)\|$.

What is $C(u)$? One of the terms it contains is

$$\|\mathrm{div}\, \partial_{tt} u(t)\|_{H^1(\Omega)}$$

$$\|\pi_h^0 p(t) - p_h^*(t)\|_{L^2(\Omega)} \leq Ch^2 \left(\|\partial_t u(t)\|_{H^1(\Omega)} + \|\mathrm{div}\, \partial_t u(t)\|_{H^1(\Omega)}\right)$$

A new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h.
\end{aligned}
$$

Theorem (Error estimate for the semi-discretization)

$$
\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} \le C(u)h^2
$$

Proof: Split $\|\pi_h^0 p(t) - p_h(t)\| \le \|\pi_h^0 p(t) - p_h^*(t)\| + \|p_h^*(t) - p_h(t)\|$.

What is $C(u)$? One of the terms it contains is

$$
\|\operatorname{div} \partial_{tt} u(t)\|_{H^1(\Omega)}
$$

$$
\|\pi_h^0 \partial_t p(t) - \partial_t p_h^*(t)\|_{L^2(\Omega)} \le Ch^2 \left(\|\partial_t \partial_t u(t)\|_{H^1(\Omega)} + \|\operatorname{div} \partial_t \partial_t u(t)\|_{H^1(\Omega)}\right)
$$

Post-processing for the pressure

Idea : Construct $\widetilde{p}_h \in P_1(\mathcal{T}_h)$ from u_h, p_h. Testing the momentum equation with $\nabla q \in L^2(\Omega)^2$ gives

$$(\nabla p, \nabla q)_K = -(\partial_t u, \nabla q)_K.$$

Post-processing for the pressure

Idea : Construct $\widetilde{p}_h \in P_1(\mathcal{T}_h)$ from u_h, p_h. Testing the momentum equation with $\nabla q \in L^2(\Omega)^2$ gives

$$(\nabla p, \nabla q)_K = -(\partial_t u, \nabla q)_K.$$

Problem

For all $K \in \mathcal{T}_h$, $t > 0$ find $\widetilde{p}_h(t) \in P_1(K)$ such that

$$\begin{aligned} (\nabla \widetilde{p}_h(t), \nabla \widetilde{q}_h)_K &= -(\partial_t u_h(t), \nabla \widetilde{q}_h)_K \quad && \forall \widetilde{q}_h \in P_1(K) \\ (\widetilde{p}_h(t), q_h)_K &= (p_h(t), q_h)_K && \forall q_h \in P_0(K), \end{aligned}$$

- R. Stenberg *Postprocessing schemes for some mixed finite elements*. RAIRO Model. Math. Anal. Numer. 1991
- Y. Chen *Global superconvergence for a mixed finite element method for the wave equation*. Systems Sci. Math. Sci. 1999

Post-Processing for the pressure

Theorem

For (p, u) sufficiently smooth, we have

$$\|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq C(p, u)h^2$$

Post-Processing for the pressure

Theorem

For (p, u) sufficiently smooth, we have

$$\|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq C(p, u)h^2$$

We split the error

$$\begin{aligned}
\|p - \widetilde{p}_h\|_{L^2(K)} &\leq \|p - \pi_1 p\|_{L^2(K)} + \|\pi_0(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)} + \|(\mathrm{Id} - \pi_0)(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)}. \\
&\leq \|p - \pi_1 p\|_{L^2(K)} + \|\pi_0 p - p_h\|_{L^2(K)} + h_K \|\nabla(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)}.
\end{aligned}$$

Post-Processing for the pressure

Theorem

For (p, u) sufficiently smooth, we have

$$\|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq C(p, u)h^2$$

We split the error

$$\begin{aligned}\|p - \widetilde{p}_h\|_{L^2(K)} &\leq \|p - \pi_1 p\|_{L^2(K)} + \|\pi_0(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)} + \|(\mathrm{Id} - \pi_0)(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)}. \\ &\leq \|p - \pi_1 p\|_{L^2(K)} + \|\pi_0 p - p_h\|_{L^2(K)} + h_K \|\nabla(\pi_1 p - \widetilde{p}_h)\|_{L^2(K)}.\end{aligned}$$

We compute

$$\begin{aligned}(\nabla(\pi_1 p - \widetilde{p}_h), \nabla\widetilde{q}_h)_K &= (\nabla(\pi_1 p - p), \nabla\widetilde{q}_h)_K + (\nabla(p - \widetilde{p}_h), \nabla\widetilde{q}_h)_K \\ &= (\nabla(\pi_1 p - p), \nabla\widetilde{q}_h)_K - (\partial_t(u - u_h), \nabla\widetilde{q}_h)_K \\ &\leq \left(\|\nabla(\pi_1 p - p)\|_{L^2(K)} + \|\partial_t(u - u_h)\|_{L^2(K)}\right) \|\nabla\widetilde{q}_h\|_{L^2(K)}.\end{aligned}$$

Auxiliary functions

Consider the functions $u_h^* \in C^1(0,T;V_h)$ and $p_h^* \in C(0,T;Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \mathrm{div}\; v_h) &= 0 && \forall v_h \in V_h \\
(\mathrm{div}\; \partial_t u_h^*(t), q_h) &= (\mathrm{div}\; \partial_t u(t), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

and

$$
\begin{aligned}
(u_h^*(0), v_h)_h - (r_h^*(0), \mathrm{div}\; v_h) &= (u(0), v_h) && \forall v_h \in V_h \\
(\mathrm{div}\; u_h^*(0), q_h) &= (\mathrm{div}\; u(0), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

Lemma (Approximation error estimates)

We have

$$
(\mathrm{div}\; u_h^*(t) - \mathrm{div}\; u(t), q_h) = 0, \quad \forall q_h \in Q_h
$$

Moreover, if Ω is convex, we have

$$
\|\pi_h^0 p(t) - p_h^*(t)\|_{L^2(\Omega)} \le Ch^2 \left(\|\partial_t u(t)\|_{H^1(\Omega)} + \|\mathrm{div}\; \partial_t u(t)\|_{H^1(\Omega)} \right)
$$

whenever u is sufficiently smooth.

Auxiliary functions

Consider the functions $\widetilde{u}_h^* \in C^1(0, T; V_h)$ and $\widetilde{p}_h^* \in C(0, T; Q_h)$ satisfying

$$
\begin{aligned}
(\partial_t \widetilde{u}_h^*(t), v_h) - (\widetilde{p}_h^*(t), \mathrm{div}\ v_h) &= 0 && \forall v_h \in V_h \\
(\mathrm{div}\ \partial_t \widetilde{u}_h^*(t), q_h) &= (\mathrm{div}\ \partial_t u(t), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

and

$$
\begin{aligned}
(\widetilde{u}_h^*(0), v_h) - (\widetilde{r}_h^*(0), \mathrm{div}\ v_h) &= (u(0), v_h) && \forall v_h \in V_h \\
(\mathrm{div}\ \widetilde{u}_h^*(0), q_h) &= (\mathrm{div}\ u(0), q_h) && \forall q_h \in Q_h
\end{aligned}
$$

Lemma (Approximation error estimates)

We have

$$(\mathrm{div}\ \widetilde{u}_h^*(t) - \mathrm{div}\ u(t), q_h) = 0, \quad \forall q_h \in Q_h$$

Moreover, we get

$$\|u(t) - \widetilde{u}_h^*(t)\| \le C(u)h^2$$

whenever u is sufficiently smooth.

Post-processing for the velocity

First try :

$$(\widetilde{u}_h(t), v_h) = (u_h(t), v_h)_h \qquad \forall v_h \in V_h$$

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2} \leq \overbrace{\|u(t) - \pi_1^0 u(t)\|_{L^2}}^{O(h^2)} + \overbrace{\|\pi_1^0 u(t) - \widetilde{u}_h(t)\|_{L^2}}^{O(h^{3/2})} \leq C(u)h^{3/2} \quad ?$$

Post-processing for the velocity

First try :

$$(\widetilde{u}_h(t), v_h) = (u_h(t), v_h)_h \qquad \forall v_h \in V_h$$

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2} \leq \overbrace{\|u(t) - \pi_1^0 u(t)\|_{L^2}}^{O(h^2)} + \overbrace{\|\pi_1^0 u(t) - \widetilde{u}_h(t)\|_{L^2}}^{O(h^{3/2})} \leq C(u)h^{3/2} \quad ?$$

Second try :

$$\begin{aligned} (\widetilde{u}_h(t), v_h) - (\widetilde{r}_h(t), \mathrm{div}\ v_h) &= (u_h(t), v_h)_h \qquad && \forall v_h \in V_h, \\ (\mathrm{div}\ \widetilde{u}_h(t), q_h) &= (\mathrm{div}\ u_h(t), q_h) \qquad && \forall q_h \in Q_h. \end{aligned}$$

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2} \leq \overbrace{\|u(t) - \pi_1^0 u(t)\|_{L^2}}^{O(h^2)} + \overbrace{\|\pi_1^0 u(t) - \widetilde{u}_h(t)\|_{L^2}}^{O(h^2)} \leq C(u)h^2$$

Post-processing for the velocity

Problem (Post-processing strategy for the velocity)

For every $0 \leq t \leq T$*, find* $\widetilde{u}_h(t) \in V_h$ *such that*

$$
\begin{aligned}
(\widetilde{u}_h(t), v_h) - (\widetilde{r}_h(t), \mathrm{div}\ v_h) &= (u_h(t), v_h)_h \qquad && \forall v_h \in V_h, \\
(\mathrm{div}\ \widetilde{u}_h(t), q_h) &= (\mathrm{div}\ u_h(t), q_h) \qquad && \forall q_h \in Q_h.
\end{aligned}
$$

Post-processing for the velocity

Problem (Post-processing strategy for the velocity)

For every $0 \le t \le T$*, find* $\widetilde{u}_h(t) \in V_h$ *such that*

$$\begin{aligned} (\widetilde{u}_h(t), v_h) - (\widetilde{r}_h(t), \mathrm{div}\ v_h) &= (u_h(t), v_h)_h \qquad && \forall v_h \in V_h, \\ (\mathrm{div}\ \widetilde{u}_h(t), q_h) &= (\mathrm{div}\ u_h(t), q_h) && \forall q_h \in Q_h. \end{aligned}$$

Theorem (Error estimate for the improved velocity)

Let Ω *be convex. Then*

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} \le C(u)h^2$$

Post-processing for the velocity

Proof:
Post-processing scheme at $t = 0$

$$
\begin{aligned}
(\widetilde{u}_h(0), v_h) - (\widetilde{r}_h(0), \operatorname{div}\ v_h) &= (u_h(0), v_h)_h \qquad && \forall v_h \in V_h,\\
(\operatorname{div}\ \widetilde{u}_h(0), q_h) &= (\operatorname{div}\ u_h(0), q_h) \qquad && \forall q_h \in Q_h.
\end{aligned}
$$

Exact elliptic projection for the initial conditions

$$
\begin{aligned}
(\widetilde{u}_h^*(0), v_h) - (\widetilde{r}_h^*(0), \operatorname{div}\ v_h) &= (u(0), v_h) \qquad && \forall v_h \in V_h\\
(\operatorname{div}\ \widetilde{u}_h^*(0), q_h) &= (\operatorname{div}\ u(0), q_h) \qquad && \forall q_h \in Q_h
\end{aligned}
$$

Inexact elliptic projection for the initial conditions

$$
\begin{aligned}
(u_h^*(0), v_h)_h - (r_h^*(0), \operatorname{div}\ v_h) &= (u(0), v_h) \qquad && \forall v_h \in V_h\\
(\operatorname{div}\ u_h^*(0), q_h) &= (\operatorname{div}\ u(0), q_h) \qquad && \forall q_h \in Q_h
\end{aligned}
$$

Post-processing for the velocity

Proof:

$$(i) = (\widetilde{u}_h(0), v_h) - (\widetilde{r}_h(0), \operatorname{div}\ v_h) = (u_h(0), v_h)_h$$
$$(ii) = (\widetilde{u}_h^*(0), v_h) - (\widetilde{r}_h^*(0), \operatorname{div}\ v_h) = (u(0), v_h)$$
$$(iii) = (u_h^*(0), v_h)_h - (r_h^*(0), \operatorname{div}\ v_h) = (u(0), v_h)$$

Computing $(ii) - (i) - (iii)$ and using $u_h(0) = u_h^*(0)$ yields

$$(\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h) = (\widetilde{r}_h^*(0) - \widetilde{r}_h(0) - r_h^*(0), \operatorname{div}\ v_h)$$

This means

$$(\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div}\ v_h = 0$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

Post-processing scheme

$$\begin{aligned}
(\partial_t \widetilde{u}_h(t), v_h) - (\partial_t \widetilde{r}_h(t), \operatorname{div} v_h) &= (\partial_t u_h(t), v_h)_h \qquad && \forall v_h \in V_h \\
(\operatorname{div} \partial_t \widetilde{u}_h(t), q_h) &= (\operatorname{div} \partial_t u_h(t), q_h) \qquad && \forall q_h \in Q_h
\end{aligned}$$

Exact elliptic projection

$$\begin{aligned}
(\partial_t \widetilde{u}_h^*(t), v_h) - (\widetilde{p}_h^*(t), \operatorname{div} v_h) &= 0 \qquad && \forall v_h \in V_h \\
(\operatorname{div} \partial_t \widetilde{u}_h^*(t), q_h) &= (\operatorname{div} \partial_t u(t), q_h) \qquad && \forall q_h \in Q_h
\end{aligned}$$

Inexact elliptic projection

$$\begin{aligned}
(\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \operatorname{div} v_h) &= 0 \qquad && \forall v_h \in V_h \\
(\operatorname{div} \partial_t u_h^*(t), q_h) &= (\operatorname{div} \partial_t u(t), q_h) \qquad && \forall q_h \in Q_h
\end{aligned}$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

$$(i) = (\partial_t \widetilde{u}_h(t), v_h) - (\partial_t \widetilde{r}_h(t), \operatorname{div} v_h) = (\partial_t u_h(t), v_h)_h$$

$$(ii) = (\partial_t \widetilde{u}_h^*(t), v_h) - (\widetilde{p}_h^*(t), \operatorname{div} v_h) = 0$$

$$(iii) = (\partial_t u_h^*(t), v_h)_h - (p_h^*(t), \operatorname{div} v_h) = 0$$

Computing $(ii) - (i) - (iii)$ and using $\partial_t u_h(0) = \partial_t u_h^*(0)$ yields

$$(\partial_t \widetilde{u}_h^*(t) - \partial_t \widetilde{u}_h(t), v_h) = (\widetilde{p}_h^*(t) - p_h(t) - \partial_t \widetilde{r}_h(t), \operatorname{div} v_h)$$

This means

$$(\partial_t \widetilde{u}_h^*(t) - \partial_t \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(0) - \widetilde{u}_h(0), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$
$$(\partial_t \widetilde{u}_h^*(t) - \partial_t \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \mathrm{div}\ v_h = 0$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \mathrm{div}\; v_h = 0$$

Divergence condition of the post-processing scheme

$$(\mathrm{div}\; \widetilde{u}_h(t), q_h) = (\mathrm{div}\; u_h(t), q_h) \qquad \forall q_h \in Q_h$$

Property of the inexact elliptic projection

$$(\mathrm{div}\; \widetilde{u}_h^*(t), q_h) = (\mathrm{div}\; u(t), q_h) \qquad \forall q_h \in Q_h$$

Property of the exact elliptic projection

$$(\mathrm{div}\; u_h^*(t), q_h) = (\mathrm{div}\; u(t), q_h) \qquad \forall q_h \in Q_h$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

Divergence condition of the post-processing scheme

$$\begin{aligned}
(i) &= (\operatorname{div} \widetilde{u}_h(t), q_h) = (\operatorname{div} u_h(t), q_h) \\
(ii) &= (\operatorname{div} \widetilde{u}_h^*(t), q_h) = (\operatorname{div} u(t), q_h) \\
(iii) &= (\operatorname{div} u_h^*(t), q_h) = (\operatorname{div} u(t), q_h)
\end{aligned}$$

Computing $(ii) - (i) - (iii)$ yields

$$(\operatorname{div} (\widetilde{u}_h^*(t) - \widetilde{u}_h(t) + u_h(t) - u_h^*(t)), q_h) = 0 \qquad \forall q_h \in Q_h$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \ \text{ with } \ \mathrm{div}\ v_h = 0$$

$$(\mathrm{div}\,(\widetilde{u}_h^*(t) - \widetilde{u}_h(t) + u_h(t) - u_h^*(t)), q_h) = 0 \qquad \forall q_h \in Q_h$$

Post-processing for the velocity

Proof:

$$(\widetilde{u}_h^*(t) - \widetilde{u}_h(t), v_h) = 0 \qquad \forall v_h \in V_h \text{ with } \operatorname{div} v_h = 0$$

$$(\operatorname{div}(\widetilde{u}_h^*(t) - \widetilde{u}_h(t) + u_h(t) - u_h^*(t)), q_h) = 0 \qquad \forall q_h \in Q_h$$

Actual start of the proof:

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} \le \|u(t) - \widetilde{u}_h^*(t)\|_{L^2(\Omega)} + \|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)}$$

We have $\|u(t) - \widetilde{u}_h^*(t)\|_{L^2(\Omega)} \le C(u)h^2$. For the second term, we compute

$$\begin{aligned} \|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)}^2 &= (\widetilde{u}_h^*(t) - \widetilde{u}_h(t), \widetilde{u}_h^*(t) - \widetilde{u}_h(t) + u_h(t) - u_h^*(t))_\Omega \\ &\quad + (\widetilde{u}_h^*(t) - \widetilde{u}_h(t), u_h^*(t) - u_h(t))_\Omega \\ &\le \|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} \|u_h^*(t) - u_h(t)\|_{L^2(\Omega)} \end{aligned}$$

In summary, we get

$$\|\widetilde{u}_h^*(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} \le \|u_h^*(t) - u_h(t)\|_{L^2(\Omega)} \le C(u)h^2$$

Post-processing

Problem

For $(u_h(0), p_h(0)) = (\rho_h u_0, \pi_h^0 p_0)$ *and all* $t > 0$ *find* $(u_h(t), p_h(t)) \in V_h \times Q_h$ *such that*

$$\begin{aligned} (\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h, \\ (\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \qquad \forall q_h \in Q_h. \end{aligned}$$

Theorem (Error estimate for the semi-discretization)

Let $V_h = BDM_1$, $Q_h = P_0$. *Then if* (u, p) *sufficiently smooth*

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + \|p(t) - p_h(t)\|_{L^2(\Omega)} \leq C(u, p)h.$$

Theorem (Full post-processing error)

Let $V_h = BDM_1$, $Q_h = P_0$. *Then if* (u, p) *sufficiently smooth*

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} + \|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} \leq C(u, p)h^2.$$

The fully discrete scheme

Problem (Fully discrete problem)

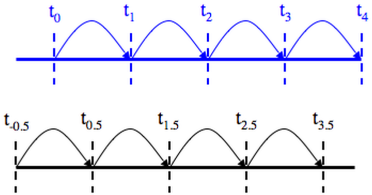
Set $u_h^0 = u_h^*(0)$ *and* $p_h^0 = p_h^*(0)$ *and define* $u_h^{-1/2} \in V_h$ *as solution of*

$$(u_h^{-1/2}, v_h)_h = (u_h^0, v_h)_h - \frac{\tau}{2}(p_h^0, \mathrm{div}\ v_h) \qquad \forall v_h \in V_h. \tag{1}$$

Then for $n \geq 0$ *find* $(u_h^{n+1/2}, p_h^{n+1}) \in V_h \times Q_h$*, such that*

$$\left(\frac{u_h^{n+1/2} - u_h^{n-1/2}}{\tau}, v_h\right)_h - (p_h^n, \mathrm{div}\ v_h) = 0 \qquad \forall v_h \in V_h. \tag{2}$$

$$\left(\frac{p_h^{n+1} - p_h^n}{\tau}, q_h\right) + (\mathrm{div}\ u_h^{n+1/2}, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{3}$$



Fully discrete results

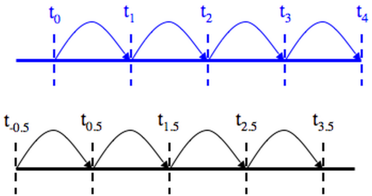
Theorem (Estimates for the discrete error)

Let Ω be convex. Then, if the CFL condition is satisfied

$$\|\widehat{u}_h^n - \widehat{u}_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - \widehat{p}_h^*(t^n)\|_{L^2(\Omega)} \leq C(u)(h^2 + \tau^2)$$

for all $0 \leq n \leq N-1$*. In addition, we also have*

$$\|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)} \leq C(u)(h^2 + \tau^2).$$



Fully discrete results

Theorem (Estimates for the discrete error)

Let Ω be convex. Then, if the CFL condition is satisfied

$$\|\widehat{u}_h^n - \widehat{u}_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - \widehat{p}_h^*(t^n)\|_{L^2(\Omega)} \leq C(u)(h^2 + \tau^2)$$

for all $0 \leq n \leq N-1$. In addition, we also have

$$\|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)} \leq C(u)(h^2 + \tau^2).$$

Theorem (Post-processing error)

Let $V_h = BDM_1$, $Q_h = P_0$. Then if (u, p) sufficiently smooth

$$\|u(t^n) - \widetilde{u}_h^n\|_{L^2(\Omega)} + \|p(t^n) - \widetilde{p}_h^n\|_{L^2(\Omega)} \leq C(u, p)(h^2 + \tau^2).$$

Remarks

- Extension to the a fully discrete scheme
- Only for lowest order $Q_h = P_0$ and $V_h = BDM_1$
- The convexity condition is sufficient.
- We require quasi-uniformity of the triangulation $\mathcal{T}_h$.
- Can compete with FDTD, does not require uniform grid.

Numerical results

Let $\Omega = (-1,1)^2$ and take

$$p(x,y,t) = \sin(\pi x)\sin(\pi y)\left(\sin\left(\pi t\sqrt{2}\right) + \cos\left(\pi t\sqrt{2}\right)\right)$$

$$u(x,y,t) = -\frac{\sqrt{2}}{2}\left(\sin\left(\pi t\sqrt{2}\right) - \cos\left(\pi t\sqrt{2}\right)\right)\begin{pmatrix}\cos(\pi x)\sin(\pi y)\\ \sin(\pi x)\cos(\pi y)\end{pmatrix}$$

Numerical results

Let $\Omega = (-1,1)^2$ and take

$$p(x,y,t) = \sin(\pi x)\sin(\pi y)\left(\sin\left(\pi t\sqrt{2}\right) + \cos\left(\pi t\sqrt{2}\right)\right)$$

$$u(x,y,t) = -\frac{\sqrt{2}}{2}\left(\sin\left(\pi t\sqrt{2}\right) - \cos\left(\pi t\sqrt{2}\right)\right)\begin{pmatrix}\cos(\pi x)\sin(\pi y)\\ \sin(\pi x)\cos(\pi y)\end{pmatrix}$$

$(\cdot,\cdot)$	$p_h^0 = \pi_h^0 p(0)$, $u_h^0 = \rho_h u(0)$
$\Vert u_h^n - u(t^n)\Vert_{L^2(\Omega)} + \Vert p_h^n - p(t^n)\Vert_{L^2(\Omega)}$	$O(h)$
$\Vert u_h^n - \rho_h u(t^n)\Vert_{L^2(\Omega)} + \Vert p_h^n - \pi_h^0 p(t^n)\Vert_{L^2(\Omega)}$	$O(h^2)$
$\Vert p(t^n) - \widetilde{p}_h^n\Vert_{L^2(\Omega)}$	$O(h^2)$

Numerical results

$(\cdot,\cdot)_h$	$p_h^0 = p_h^*(0)$, $u_h^0 = u_h^*(0)$	$p_h^0 = \pi_h^0 p(0)$, $u_h^0 = \rho_h u(0)$
$\|u_h^n - u(t^n)\|_{L^2(\Omega)} + \|p_h^n - p(t^n)\|_{L^2(\Omega)}$	$O(h)$	$O(h)$
$\|u_h^n - u_h^*(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$
$\|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h^2)$?
$\|u_h^n - u_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - p_h^*(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$
$\|p(t^n) - \widetilde{p}_h^n\|_{L^2(\Omega)}$	$O(h^2)$	$O(h^2)$?
$\|u(t^n) - \widetilde{u}_h^n\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$

Numerical results

$(\cdot,\cdot)_h$	$p_h^0 = p_h^*(0)$, $u_h^0 = u_h^*(0)$	$p_h^0 = \pi_h^0 p(0)$, $u_h^0 = \rho_h u(0)$
$\|u_h^n - u(t^n)\|_{L^2(\Omega)} + \|p_h^n - p(t^n)\|_{L^2(\Omega)}$	$O(h)$	$O(h)$
$\|u_h^n - u_h^*(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$
$\|p_h^n - \pi_h^0 p(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h^2)$?
$\|u_h^n - u_h^*(t^n)\|_{L^2(\Omega)} + \|p_h^n - p_h^*(t^n)\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$
$\|p(t^n) - \widetilde{p}_h^n\|_{L^2(\Omega)}$	$O(h^2)$	$O(h^2)$?
$\|u(t^n) - \widetilde{u}_h^n\|_{L^2(\Omega)}$	$O(h^2)$	$O(h)$

Thank you for your attention

Acknowledgement : The work of Bogdan Radu is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universität Darmstadt