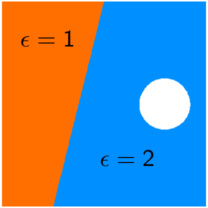
Mass-lumped mixed finite element methods for Maxwell's equations

H. Egger and B. Radu

September 12, 2019

Maxwell's equations

Application. Electromagnetic wave propagation in linear and non-dispersive but possibly inhomogeneous and anisotropic media



$$\epsilon \partial_t E(t) = \quad \mathrm{curl}\, H(t) - \sigma E(t)$$
$$\mu \partial_t H(t) = -\mathrm{curl}\, E(t)$$

in Ω, $t > 0$ with $E(0) = 0$ in Ω and $n \times E(t) = 0$ on $\partial\Omega$

Goal: systematic and flexible space discretization

- stable: no artificial energy production
- accurate: provable convergence rates
- efficient: appropriate for explicit time-stepping methods

Methods: FDTD/FIT, FVM, **FEM**, DG, ...

Galerkin approximation

Approximation spaces: $V_h \subset H_0(\text{curl};\Omega)$ and $Q_h \subset L^2(\Omega)$

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$$(\mu \partial_t H_h(t), q_h)_\Omega + (\text{curl } E_h(t), q_h)_\Omega = 0$$

for all test functions $v_h \in V_h$ and $q_h \in Q_h$, and for all $t > 0$

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Algebraic realization.

$$
\begin{aligned}
\mathbf{M}_\epsilon \partial_t \mathbf{e}(t) - \mathbf{C}^\top \mathbf{h}(t) &= 0 \\
\mathbf{D}_\mu \partial_t \mathbf{h}(t) + \mathbf{C}\ \mathbf{e}(t) &= 0
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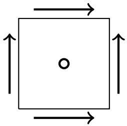
$$

Mixed finite element approximation

Finite element spaces on reference elements.



$$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q}) \qquad \phi_1 = (1-y, 0) \quad \phi_3 = (0, 1-x)$$
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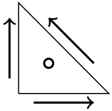

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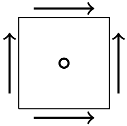

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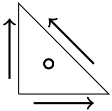

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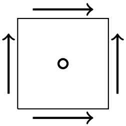

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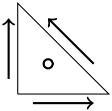

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Note: Construction for "physical" elements Q or T by Piola-transform

Lemma (accuracy) [EggerRadu'18,DupontKeenan'98,LiBank'18].
If E and H are sufficiently smooth. Then

$$\|E(t) - E_h(t)\| + \|H(t) - H_h(t)\| \leq Ch$$

By duality argument, one can show super-convergence **(ONLY 2D)**

$$\|\Pi_h^0 H(t) - H_h(t)\| \leq Ch^2$$

Mixed finite element approximation

Stability and accuracy.

Lowest order MFEM yields stable and accurate approximation in space.

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Numerical solution. Time integration of resulting ode system

$$\begin{aligned} \mathbf{M}_\epsilon \partial_t \mathbf{e}(t) - \mathbf{C}^\top \mathbf{h}(t) &= 0 \\ \mathbf{D}_\mu \partial_t \mathbf{h}(t) + \mathbf{C}\,\mathbf{e}(t) &= 0 \end{aligned}$$

by explicit schemes requires application of $\mathbf{M}_\epsilon^{-1}$ and $\mathbf{D}_\mu^{-1}$.

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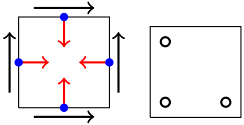
Remedy – Mass-lumping: replace $\mathbf{M}_\epsilon$ by approximation $\mathbf{M}_\epsilon^L$ such that

- $\mathbf{M}_\epsilon^L$ corresponds to positive definite matrix (stability)
- $\mathbf{M}_\epsilon^L$ is good approximation for $\mathbf{M}_\epsilon$ (accuracy)
- $(\mathbf{M}_\epsilon^L)^{-1}$ can be applied efficiently (efficiency)

construction of $\mathbf{M}_\epsilon^L$ usually via numerical quadrature; see **[Cohen'02]**.

Extended finite element space.

Add additional interior basis functions **[ElmkiesJoly'93]**.



$$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q}) \oplus B = \mathrm{EJ}_1(\widehat{Q})$$
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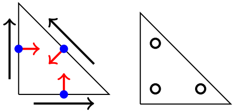

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Numerical Integration

Use the midpoint rule, which is exact for P_2 functions.

Proposed solutions

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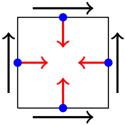
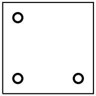

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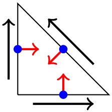
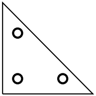

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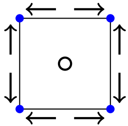
Lemma (accuracy)
If E and H are sufficiently smooth. Then

$$\|E(t) - E_h(t)\| + \|H(t) - H_h(t)\| \leq Ch$$

A different solution

Another choice of finite elements



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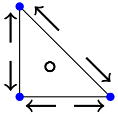

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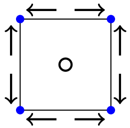

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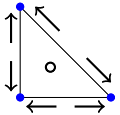

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Moreover, $\|\Pi_h^0 H(t) - H_h(t)\| \leq Ch^2$ and for a special projection $\widehat{\Pi}_h$

$$\|\widehat{\Pi}_h E(t) - E_h(t)\| \leq Ch^2$$

Idea why

Recall the lumped Galerkin discretization

$$
\begin{aligned}
(\epsilon\partial_t E_h(t), v_h)_h - (H_h(t), \text{curl } v_h)_\Omega &= 0\\
(\mu\partial_t H_h(t), q_h)_\Omega + (\text{curl } E_h(t), q_h)_\Omega &= 0
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$$

Use the elliptic projection

$$
\begin{aligned}
(\widehat{\Pi}_h\epsilon\partial_t E(t), v_h)_h - (\widehat{\pi}_h H(t), \text{curl}\, v_h) &= 0 && \forall v_h \in V_h,\\
(\text{curl}\, \widehat{\Pi}_h\partial_t E(t), q_h) &= (\text{curl}\, \partial_t E(t), q_h) && \forall q_h \in Q_h.
\end{aligned}
$$

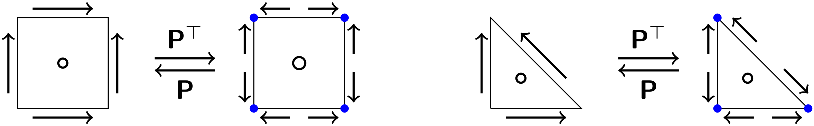
Then, we have

$$
\begin{aligned}
(\epsilon\partial_t E_h(t) - \widehat{\Pi}_h\epsilon\partial_t E(t), v_h)_h - (H_h(t) - \widehat{\pi}_h H(t), \text{curl } v_h)_\Omega &= 0\\
(\mu\partial_t H_h(t) - \widehat{\pi}_h\mu H(t), q_h)_\Omega + (\text{curl } (E_h(t) - \widehat{\Pi}_h E(t)), q_h)_\Omega &=\\
&= (\mu\partial_t H_h(t) - \widehat{\Pi}^0_h\mu H(t), q_h)_\Omega
\end{aligned}
$$

With this, we can devise a non-local post-processing strategy for E

Embedding quadrature

Idea. Use lowest order space V_h to represent solution, compute update in enriched space $\widetilde{V}_h$, and then project back to V_h



Formal representation of inverse mass matrix.

$$(\mathbf{M}_\epsilon^L)^{-1} = \mathbf{P}\,(\widetilde{\mathbf{M}}_\epsilon^L)\,\mathbf{P}^\top$$

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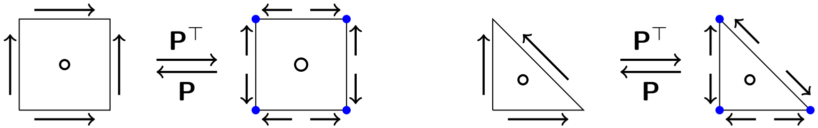

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Lemma [EggerRadu'18].

$$\|E(t) - E_h(t)\| + \|H(t) - H_h(t)\| \leq Ch$$

and superconvergence

$$\|\Pi_h^0 H(t) - H_h(t)\| \leq Ch^2 \quad \text{and} \quad \|\Pi_h E(t) - E_h(t)\| \leq ch^\gamma$$

with $\gamma = 2$ on regular and $3/2 \leq \gamma \leq 2$ on uniformly refined meshes
Moreover, method equivalent to FDTD/FIT on rectangular grids!

Higher order

For first order elements, the quadrature formula was exact for $V_h \times P_0 \subseteq P_2$. Intuition suggests that for higher order elements, it should be exact for $V_h \times P_1$. We show that it is possible to relax this condition.

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Recall the elliptic projection

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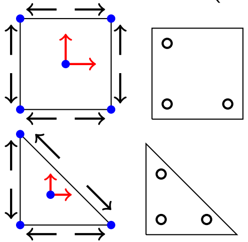

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Lemma [EggerRadu'18].

$$\|E(t) - E_h(t)\| + \|\Pi_h^0 H(t) - H_h(t)\| \leq Ch^2$$

and, under elliptic regularity

$$\|\Pi_h^0 H(t) - H_h(t)\| \leq Ch^3$$

Remark. This result also holds for Maxwells equations.

Comparison

CFL condition

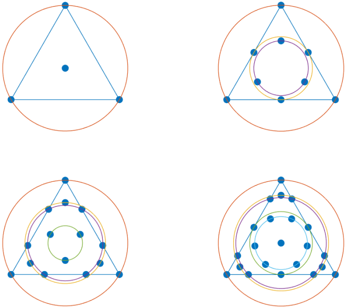
	CFL Number
$\mathcal{N}_1^{II}$-$\mathbf{P}_0$ ($\mathcal{N}_1^{I}$-$\mathbf{P}_0$)	0.288
EJ_1-$\mathbf{P}_1$	0.131
$\mathcal{N}_2^{I}$-$\mathbf{P}_1$	0.094
EJ_2-$\mathbf{P}_2^+$	0.039

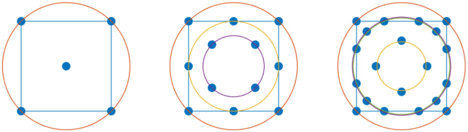
CPU Runtime

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
$\mathcal{N}_1^{II}$-$\mathbf{P}_0$	0.0590	0.0704	0.1785	0.6780	2.6101
EJ_1-$\mathbf{P}_1$	0.1032	0.1837	0.6682	2.4143	9.9358
$\mathcal{N}_2^{I}$-$\mathbf{P}_1$	0.1065	0.1965	0.7327	2.8115	11.4102
EJ_2-$\mathbf{P}_2^+$	0.3103	0.8093	3.4265	19.4949	86.3967

Table: Duration of 10000 time steps in seconds for lumped methods. Leap-frog with pre-computed inverse of the mass matrix.

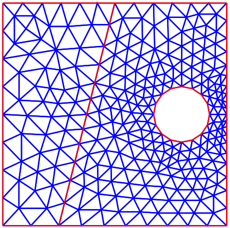
Even higher orders





Further results and open problems

Simulation on unstructured mesh



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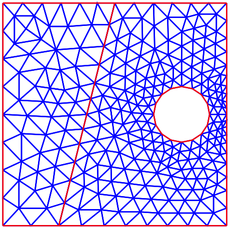

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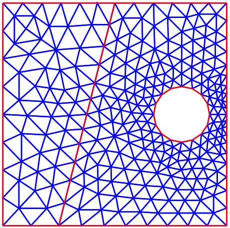

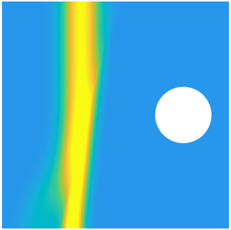

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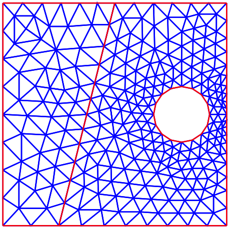

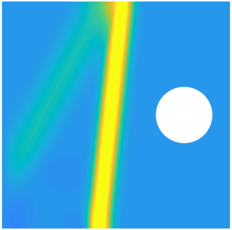

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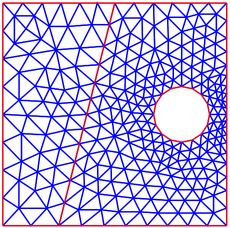

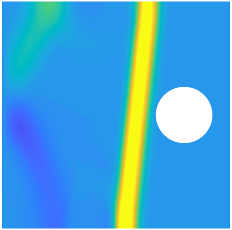

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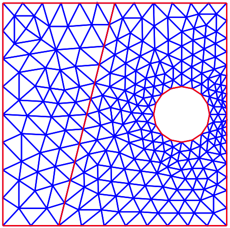

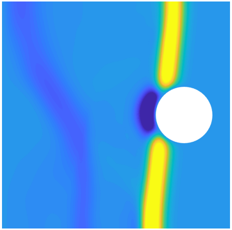

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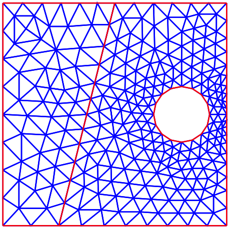
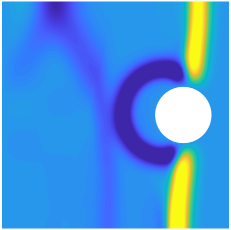
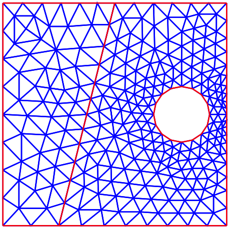
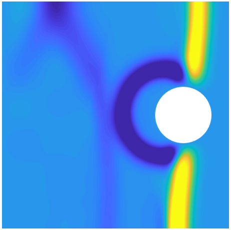




Further results and open problems

Simulation on unstructured mesh



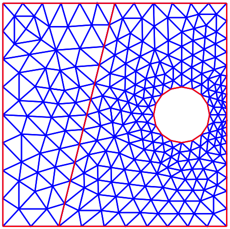


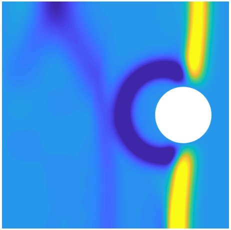
Further results and current research

- similar results for tetrahedra, hexahedra, prisms
- quadrature rules and mass lumping for higher order approximations
- local post-processing schemes yielding h^2 full convergence

Further results and open problems

Simulation on unstructured mesh





Further results and current research

- similar results for tetrahedra, hexahedra, prisms
- quadrature rules and mass lumping for higher order approximations
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Further reseearch directions

- complete error analysis for uniformly refined unstructured grids
- finite elements, quadrature rules, and mass lumping for pyramids

Some references

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