A mixed finite element method for nonlinear magnetostatics

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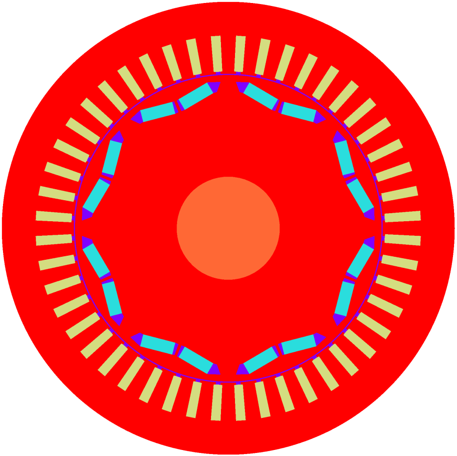

Figure: Cross section of a permanent magnet synchronous machine (PMSM)

Relevance:

- Automotive industry
- Parameter studies, shape optimization of motors
- Torque computation of electric motors

Motivation

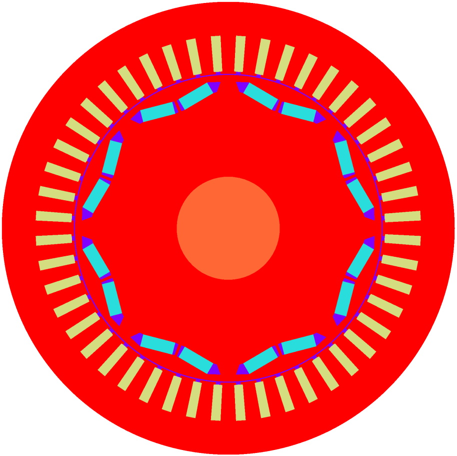

Figure: Cross section of a permanent magnet synchronous machine (PMSM)

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Challenges:

- Anisotropic nonlinear material laws
- Rotating geometries
- Non-smooth coefficients
- Fast, reliable, accurate simulations

Maxwell's equations for magnetostatics

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- Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain and consider Maxwell's equations

$$\begin{aligned} \operatorname{curl} H &= J && \text{(Ampere Law)} \\ \operatorname{div} B &= 0 && \text{(Gauss Law)} \\ n \cdot B &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Nonlinear material law

$$H = f'(B) \qquad \text{or} \qquad B = g'(H)$$

Here f and g are the energy/coenergy densities.

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- Nonlinear material law

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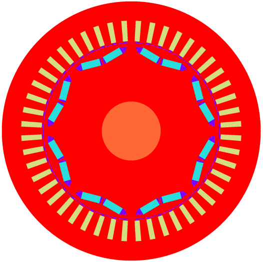
Here f and g are the energy/coenergy densities. Further, we have



- Coil, air and shaft : $g'(H) = \mu H$
- Magnets : $g'(H) = \widetilde{\mu} H - M$
- Iron : $g'(H)$

The primal formulation

- Functional analytical setting: For $v = (v_1, v_2)$ and q, let

$$\operatorname{curl} v = \partial_x v_2 - \partial_y v_1 \qquad \operatorname{Curl} q = (\partial_y q, -\partial_x q)^\top$$

$$H(\operatorname{curl}, \Omega) = \{v \in L^2(\Omega)^2 \mid \operatorname{curl} v \in L^2(\Omega)\}$$

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- For a simply connected Ω, we know there exists A with $\operatorname{Curl} A = B$.
- Substituting leads to

$$\begin{aligned} \operatorname{curl}(f'(\operatorname{Curl} A)) &= J \qquad \text{in } \Omega \\ A &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

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$$\begin{aligned} \operatorname{curl}(f'(\operatorname{Curl} A)) &= J && \text{in } \Omega \\ A &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Weak formulation:

> Find $A \in H_0(\operatorname{Curl}, \Omega)$:
>
> $$(f'(\operatorname{Curl} A), \operatorname{Curl} v) = (J, v) \quad \forall v \in H_0(\operatorname{Curl}, \Omega)$$

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Find $A \in H_0(\mathrm{Curl}, \Omega)$:

$$(f'(\mathrm{Curl}\, A), \mathrm{Curl}\, v) = (J, v) \quad \forall v \in H_0(\mathrm{Curl}, \Omega)$$

Theorem

Assume f satisfies:

- $f \in C^1(\Omega)$
- f' *Lipschitz continuous*
- f' *is strongly monotonic (coercive), i.e.* $(f'(x) - f'(y))(x - y) \geq c|x - y|^2$

Then the system above admits a unique solution.

- 1960 - Zarantonello - Solving functional equations by contractive averaging.

The primal discretization

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Find $A_h \in V_h \subset H(\mathrm{Curl}, \Omega)$:

$$(f'(\mathrm{Curl}\, A_h), \mathrm{Curl}\, v_h) = (J, v_h) \quad \forall v_h \in V_h \subseteq H_0(\mathrm{Curl}, \Omega)$$

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- The diescrete problem admits a unique solution as well.

The primal discretization

- Remarks:
  - Used as a standard in solving magnetostatics. (FEMM)
  - Linearizing with Newton, each step requires the solution of an elliptic problem.
  - Line search for the equivalent minimization problem $\min_{A \in H_0^1(\Omega)} \int_\Omega f(\operatorname{curl} A)$
  - Provable convergence orders

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- But ...
  - If $V_h = P_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, then B and H are only approximated in $P_0(\mathcal{T}_h)$, which are actually the quantities of interest, in general.
  - The physical Ampere law is only satisfied in a weak sense, while the material law is satisfied pointwise on the discrete level.

$$\operatorname{curl}(\overbrace{f'(\underbrace{\operatorname{Curl} A}_{B})}^{H}) = J$$

  - We do not have access to either $t \cdot H$ or $n \cdot B$ across element interfaces.

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  - Used as a standard in solving magnetostatics. (FEMM)
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- Idea: eliminate B instead of H. This leads to:

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- Variational formulation

Find $H \in H(\operatorname{curl}, \Omega)$ and $A \in L^2(\Omega)$:

$$\begin{aligned} (g'(H), v) - (A, \operatorname{curl} v) &= 0 && \forall v \in H(\operatorname{curl}, \Omega) \\ (\operatorname{curl} H, q) &= (J, q) && \forall q \in L^2(\Omega) \end{aligned}$$

- 1992 - Barba, Marini, Savini - Mixed finite elements in magnetostatics.

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- The solution of the variational formulation solves

$$\min_{H \in H(\operatorname{curl}, \Omega)} \int_\Omega g(H) \quad \text{s.t.} \quad \operatorname{curl} H = J$$

- Variational formulation

Find $H \in H(\mathrm{curl}, \Omega)$ and $A \in L^2(\Omega)$:

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- g' *is strongly monotonic (coercive), i.e.* $(g'(x) - g'(y))(x - y) \geq c|x - y|^2$

Then the system above admits a unique solution.

- Discrete variational formulation

Find $H_h \in V_h \subseteq H(\mathrm{curl}, \Omega)$ and $A_h \in Q_h \subseteq L^2(\Omega)$:

$$\begin{aligned} (g'(H_h), v_h) - (A_h, \mathrm{curl}\, v_h) &= 0 && \forall v_h \in V_h \\ (\mathrm{curl}\, H_h, q_h) &= (J, q_h) && \forall q_h \in Q_h \end{aligned}$$

Theorem

Assume g satisfies:

- $g \in C^1(\Omega)$
- g' *Lipschitz continuous*
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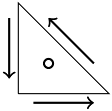
Further assume (V_h, Q_h) form a stable inf-sup pair.

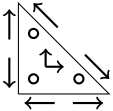
Then the system above admits a unique solution.

Finite element spaces

- Finite element spaces on reference elements.



$$V_h(T) = \mathcal{N}_0(T)$$
$$Q_h(T) = P_0(T)$$



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- 1980 - Nedelec - Mixed finite elements in $\mathbb{R}^3$

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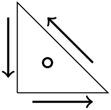

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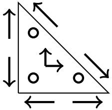

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Theorem

For $V_h = \mathcal{N}_k(\mathcal{T}_h) \cap H(\mathrm{curl}, \Omega)$ and $Q_h = P_k(\mathcal{T}_h)$, we have

$$\|H - H_h\|_{H(\mathrm{curl})} + \|A - A_h\|_{L^2} \leq Ch^k$$

where (H_h, A_h) and (H, A) solve the continuous and discrete problems, resp.

- 1992 - Monk - Analysis of a finite element method for Maxwell's equations
- 1993 - Monk - An analysis of Nedelec's method for spatial discretization of Maxwell's equations

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- We use the Newton method: Construct sequences $(H_h^n, A_h^n)_{n\geq 1}$ with

$$(H_h^{n+1}, A_h^{n+1}) = (H_h^n, A_h^n) - \tau^n(\delta H_h^n, \delta A_h^n)$$

where (H_h^n, A_h^n) solve

Find $\delta H_h^n \in V_h \subseteq H(\mathrm{curl}, \Omega)$ and $\delta A_h^n \in Q_h \subseteq L^2(\Omega)$:

$$\begin{aligned} (g''(H_h^n)\delta H_h^n, v_h) - (\delta A_h^n, \mathrm{curl}\, v_h) &= (g'(H_h^n), v_h) \qquad &\forall v_h \in V_h \\ (\mathrm{curl}\, \delta H_h^n, q_h) &= (J, q_h) \qquad &\forall q_h \in Q_h \end{aligned}$$

- Existence and uniqueness of solutions for each linearized system follows from the assumptions on g and the fact that V_h and Q_h form a stable pair in the context of Brezzi theory.

- We choose τ^n by using Armijo Backtracking. This guarantees global convergence for the Newton method.

Hybridization

- In each Newton step, we need to solve a system of the form

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Hybridization

- In each Newton step, we need to solve a system of the form
- Relax the $H(\mathrm{curl})$-continuity by using hybridization.

Find $\delta H_h^n \in V_h^{di}, \delta A_h^n \in Q_h$ and $\widehat{\delta A_h^n} \in L_h$ for all $T \in \mathcal{T}_h$:

$$
\begin{aligned}
(g''(H_h^n)\delta H_h^n, v_h)_T - (\delta A_h^n, \operatorname{curl} v_h)_T + (\widehat{\delta A_h^n}, [n \times v_h])_F &= (g'(\delta A_h^n), v_h)_T && \forall\, v_h \in V_h^{di} \\
(\operatorname{curl} \delta H_h^n, q_h)_T &= (J, q_h) && \forall\, q_h \in Q_h \\
([n \times \delta H_h^n], \mu_h)_F &= 0 && \forall\, \mu_h \in L_h
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\end{aligned}
$$

- We can now proceed to eliminate both δH_h and δA_h algebraically, which yields a symmetric positive definite system for the Lagrange multiplier $\widehat{\delta A_h}$ alone.

$$
\begin{pmatrix} \mathsf{M} & -\mathsf{B}^\top & \mathsf{L}^\top \\ \mathsf{B} & 0 & 0 \\ \mathsf{L} & 0 & 0 \end{pmatrix}
\begin{pmatrix} \delta \mathsf{h} \\ \delta \mathsf{a} \\ \delta \hat{\mathsf{a}} \end{pmatrix}
=
\begin{pmatrix} \mathsf{r} \\ \mathsf{j} \\ 0 \end{pmatrix}
$$

Hybridization

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&(g''(H_h^n)\delta H_h^n, v_h)_T - (\delta A_h^n, \mathrm{curl}\, v_h)_T + (\widehat{\delta A_h^n}, [n \times v_h])_F &&= (g'(\delta A_h^n), v_h)_T \quad &&\forall v_h \in V_h^{di}\\
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a symmetric positive definite system for the Lagrange multiplier $\widehat{\delta A_h}$ alone.

$$
\begin{pmatrix} -\mathsf{BM}^{-1}\mathsf{B}^\top & \mathsf{BM}^{-1}\mathsf{L}^\top \\ -\mathsf{LM}^{-1}\mathsf{B}^\top & \mathsf{LM}^{-1}\mathsf{L}^\top \end{pmatrix} \begin{pmatrix} \delta \mathsf{a} \\ \delta \hat{\mathsf{a}} \end{pmatrix} = \begin{pmatrix} \mathsf{BM}^{-1}\mathsf{r} - \mathsf{j} \\ \mathsf{LM}^{-1}\mathsf{r} \end{pmatrix}
$$

- In each Newton step, we need to solve a system of the form
- Relax the $H(\mathrm{curl})$-continuity by using hybridization.

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- We can now proceed to eliminate both δH_h and δA_h algebraically, which yields a symmetric positive definite system for the Lagrange multiplier $\widehat{\delta A_h}$ alone.

$$\left(-\mathsf{L}\mathsf{M}^{-1}\mathsf{L}^\top + \mathsf{L}\mathsf{M}^{-1}\mathsf{B}^\top(\mathsf{B}\mathsf{M}^{-1}\mathsf{B}^\top)^{-1}\mathsf{B}\mathsf{M}^{-1}\mathsf{L}^\top\right)\delta\widehat{\mathsf{a}} = \ldots$$

- (1) Start with initial guesses H_h^0, A_h^0.
- (2) For $n \geq 0$ assemble the system

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\end{aligned}
$$

and reduce it to a s.p.d. system in $\widehat{\delta A_h^n}$ alone.

- (3) Solve for $\widehat{\delta A_h^n}$
- (4) From $\widehat{\delta A_h^n}$, compute δH_h^n and δA_h^n from local systems
- (5) Compute the step size τ_n by Armijo backtracking and apply the update

$$(H_h^{n+1}, A_h^{n+1}) = (H_h^n, A_h^n) - \tau^n(\delta H_h^n, \delta A_h^n)$$

- (6) If tolerance is achieved, stop. Otherwise jump to (2).

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- (1) Start with initial guesses H_h^0, A_h^0.
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\end{aligned}$$

and reduce it to a s.p.d. system in $\widehat{\delta A_h^n}$ alone. This is fast!

- (3) Solve for $\widehat{\delta A_h^n}$ This is where most of the computational time is spent!
- (4) From $\widehat{\delta A_h^n}$, compute δH_h^n and δA_h^n from local systems This is fast!
- (5) Compute the step size τ_n by Armijo backtracking and apply the update

$$(H_h^{n+1}, A_h^{n+1}) = (H_h^n, A_h^n) - \tau^n(\delta H_h^n, \delta A_h^n)$$

- (6) If tolerance is achieved, stop. Otherwise jump to (2).

Postprocessing for the B-field

- While $H_h \in H(\mathrm{curl}, \Omega)$, we just have $B_h := g'(H_h) \in L^2(\Omega)$.

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- While $H_h \in H(\mathrm{curl}, \Omega)$, we just have $B_h := g'(H_h) \in L^2(\Omega)$.
- Let's look at the equations again:

$$
\begin{aligned}
B &= g'(H) \\
\operatorname{div} B &= 0 \\
B \cdot n &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Postprocessing for the B-field

- While $H_h \in H(\mathrm{curl}, \Omega)$, we just have $B_h \coloneqq g'(H_h) \in L^2(\Omega)$.
- Let's look at the equations again:

$$\begin{aligned} B + \nabla \ell &= g'(H) \\ \operatorname{div} B &= 0 \\ B \cdot n &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

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- Integration by parts

$$
\begin{aligned}
(B, w) - (\ell, \operatorname{div} w) + (\ell, n \cdot w)_\Gamma &= (g'(H), w) \\
(\operatorname{div} B, z) &= 0
\end{aligned}
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- Variational formulation

Find $B \in H_0(\mathrm{div}, \Omega)$ and $\ell \in L^2(\Omega)$:

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(B, w) - (\ell, \operatorname{div} w) &= (g'(H), w) \qquad && \forall w \in H_0(\mathrm{div}, \Omega) \\
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- Discrete variational formulation

Find $B_h \in W_h \subseteq H_0(\mathrm{div}, \Omega)$ and $\ell_h \in Z_h \subseteq L^2(\Omega)$:

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- Existence ✓, Uniqueness ✓. Can be solved efficiently by hybridization.

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- We now have both $H_h \in H(\mathrm{curl}, \Omega)$ and $B_h \in H_0(\mathrm{div}, \Omega)$

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- Existence ✓, Uniqueness ✓. Can be solved efficiently by hybridization.
- We now have both $H_h \in H(\mathrm{curl}, \Omega)$ and $B_h \in H_0(\mathrm{div}, \Omega)$
- A discrete B_h can also be constructed locally from $\widehat{A}_h$ and the fields H_h, A_h.

Why mixed? Outlook

ÖAW RICAM

- Why use the mixed formulation over the primal one?
  - We approximate the constitutive laws exactly, while the material laws are only imposed weakly.
  - Direct access to $t \cdot H$ and $n \cdot B$ along interfaces.
  - Better handling of jumping coefficients.
  - Stator-rotor coupling by mortaring is automatically covered.

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- However...
  - Larger system matrices, but better stencils.
  - Further improvement of the stencil possible through numerical integration and clever choices of basis functions.
  - Number of Newton iterations?
  - Fair time-to-solution simulations needed...