Mass-lumped finite element method for Maxwell's equations

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MS11: Numerical methods for wave propagation with applications in electromagnetics and geophysics

Maxwell's equations

Electromagnetic wave propagation in linear and non-dispersive but possibly inhomogeneous and anisotropic media

$$\begin{aligned} \varepsilon\,\partial_t E(t) &= \quad \mathrm{curl}\; H(t) - \sigma E(t) \\ \mu\,\partial_t H(t) &= -\mathrm{curl}\; E(t) \end{aligned}$$

in Ω, with $E(0) = E_0$ and $H(0) = H_0$ in Ω and $n \times E(t) = 0$ on $\partial\Omega$

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Goal: systematic and flexible space discretization

- stable: no artificial energy production
- accurate: provable convergence rates
- efficient: appropriate for explicit time-stepping methods

Methods: FDTD/FIT, FEM, FVM, DG, ...

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Approximation spaces: $V_h \subset H_0(\text{curl}; \Omega)$ and $Q_h \subset L^2(\Omega)$

Galerkin method: For $t > 0$, find $E_h(t) \in V_h$ and $H_h(t) \in Q_h$ such that

$$\begin{aligned} (\epsilon \partial_t E_h(t), v_h)_\Omega - (H_h(t), \text{curl } v_h)_\Omega &= 0 \\ (\mu \partial_t H_h(t), q_h)_\Omega + (\text{curl } E_h(t), q_h)_\Omega &= 0 \end{aligned}$$

for all test functions $v_h \in V_h$ and $q_h \in Q_h$, and for all $t > 0$.

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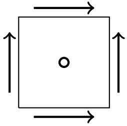
Algebraic realization.

$$\begin{aligned} \mathbf{M}_\epsilon \partial_t \mathbf{e}(t) - \mathbf{C}^\top \mathbf{h}(t) &= 0 \\ \mathbf{D}_\mu \partial_t \mathbf{h}(t) + \mathbf{C}\ \mathbf{e}(t) &= 0 \end{aligned}$$

First order elements

Finite element spaces on reference elements.



$$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q}) \qquad \phi_1 = (1-y, 0) \qquad \phi_3 = (0, 1-x)$$
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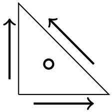

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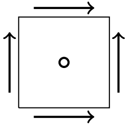

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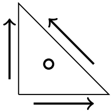

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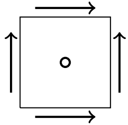

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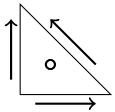

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Note: Construction for "physical" elements Q or T by Piola-transform

Lemma (accuracy) [EggerRadu'18,DupontKeenan'98,LiBank'18].
If E and H are sufficiently smooth. Then

$$\|E(t) - E_h(t)\| + \|H(t) - H_h(t)\| \le Ch$$

By duality argument, one can show super-convergence (ONLY 2D)

$$\|\Pi_h^0 H(t) - H_h(t)\| \le Ch^2$$

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Stability and accuracy.
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Remedy – Mass-lumping: replace $\mathbf{M}_\epsilon$ by approximation $\mathbf{M}_\epsilon^L$ such that

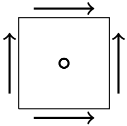
- $\mathbf{M}_\epsilon^L$ corresponds to positive definite matrix (stability)
- $\mathbf{M}_\epsilon^L$ is good approximation for $\mathbf{M}_\epsilon$ (accuracy)
- $(\mathbf{M}_\epsilon^L)^{-1}$ can be applied efficiently (efficiency)

construction of $\mathbf{M}_\epsilon^L$ usually via numerical quadrature; see **[Cohen'02]**.

Mass lumping literature

- 1990 - Lee, Madsen - A mixed FEM formulation for Maxwell's equations in the time domain
- 1995 - Cohen, Monk - Mass lumped edge elements in three dimensions
- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 3D
- 1998 - Cohen, Monk - Gauss Point Mass Lumping Schemes for Maxwell's Equations
- 1999 - Kong, Mulder, Veldhuizen - Higher-order triangular and tetrahedral finite elements with mass lumping for solving the wave equation
- 2000 - Becache, Joly, Tsogka - An analysis of new mixed finite elements for the approximation of wave propagation models
- 2001 - Mulder - Higher-order mass-lumped finite elements for the wave equation
- 2018 - Geevers, Mulder, Vegt - New higher-order mass-lumped tetrahedral elements for wave propagation modelling
- 2018 - Egger, Radu - A mass-lumped mixed finite element method for acoustic wave propagation
- 2018 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations

Observation



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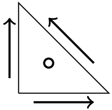

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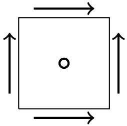

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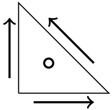

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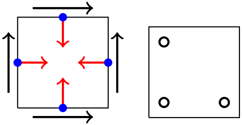
Observation: No "good" quadrature rule that leads to decoupling of entries in mass matrix for V_h.

One existing method : acute mesh lumping

- 1996 - Baranger - Connection between finite volume and mixed finite element methods

Strategy 1 : Extended finite element space

Add additional interior basis functions **[ElmkiesJoly'93]**.



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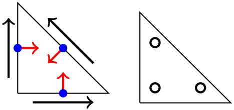

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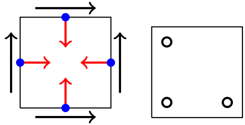

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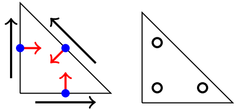

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Exactness requirement
The quadrature rule should be exact for $P_k \times V_h$, $k = 0$ for the first order case

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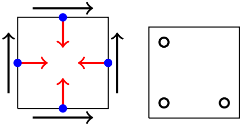

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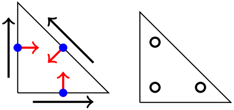

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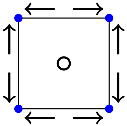
Strategy 2 : Different FEM space

Use a higher order space **[WheelerYotov'06]**



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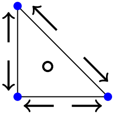

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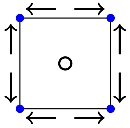

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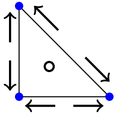

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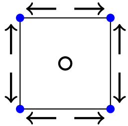

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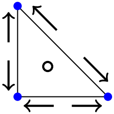

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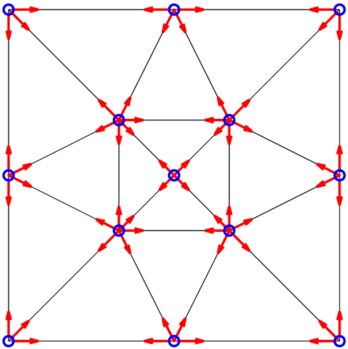

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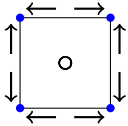

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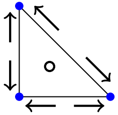

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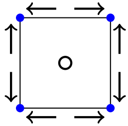

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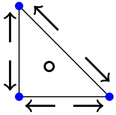

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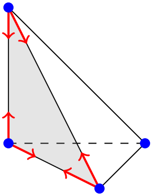
3D Same theory applies for the following element



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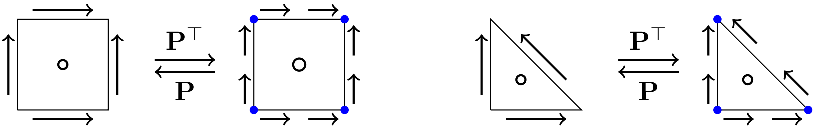

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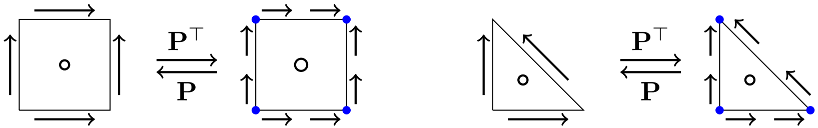

Formal representation of inverse mass matrix.

$$(\mathbf{M}_\epsilon^L)^{-1} = \mathbf{P}\,(\widetilde{\mathbf{M}}_\epsilon^L)^{-1}\,\mathbf{P}^\top$$

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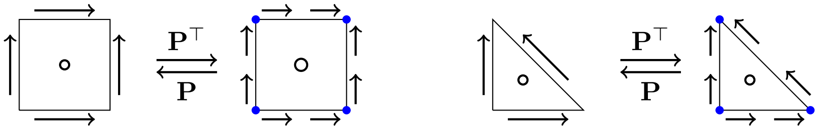
Idea : Use lowest order space V_h to represent solution, compute update in enriched space $\widetilde{V}_h$, and then project back to V_h



Formal representation of inverse mass matrix.

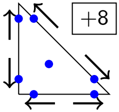
$$(\mathbf{M}_\epsilon^L)^{-1} = \mathbf{P}\ (\widetilde{\mathbf{M}}_\epsilon^L)^{-1}\ \mathbf{P}^\top$$

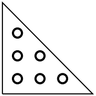
Note : The inverse is sparse, the corresponding mass matrix is full
Again: equivalence to FDTD for square elements.

- 2018 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations

Second order method

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 2D





$$V_h(\widehat{T}) = \mathcal{N}_1^I(\widehat{T}) \oplus B = \mathsf{EJ}_2(\widehat{T}) \subseteq P_3(\widehat{T})$$
$$Q_h(\widehat{T}) = P_2(\widehat{T})$$

The quadrature rule is exact for P_3 polynomials.

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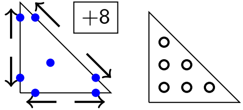

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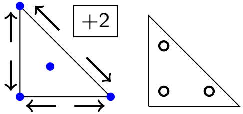
New proposal :



$$V_h(\widehat{T}) = \mathcal{N}_1^I(\widehat{T}) \subseteq P_2(\widehat{T})$$
$$Q_h(\widehat{T}) = P_1(\widehat{T})$$

The quadrature rule is exact for P_2 polynomials ... but is this enough ?.

Short notes on the analysis

Classic requirement of exactness

The quadrature rule has to be exact for $P_1(T)^d \times V_h(T)$

New requirements

(i) There exists a splitting $V_h = \widetilde{V}_h(T) \oplus W(T)$ s.t. $\dim(W(T)) = \dim(\operatorname{curl} W(T))$

(ii) The quadrature rule is exact for $P_1(T)^d \times \widetilde{V}_h(T)$

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If E and H are sufficiently smooth. Then

$$\|E(t) - E_h(t)\| + \|H(t) - H_h(t)\| \leq Ch^2$$

Second order method - 3D

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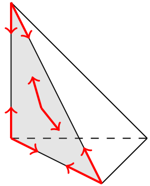

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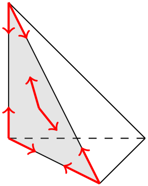

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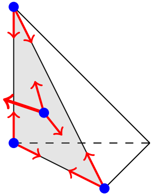

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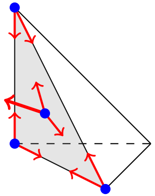

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Interior basis functions

$$\widehat{\Phi}_1 = \lambda_2\lambda_3\lambda_4\nabla\lambda_1 \qquad \widehat{\Phi}_2 = \lambda_1\lambda_3\lambda_4\nabla\lambda_2$$
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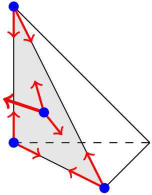

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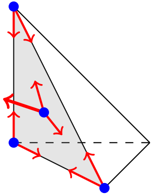

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Solution
Modify one basis function, for example $\widehat{\Phi}_4 = \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1 + 1)\nabla\lambda_4$

Second order method - 3D

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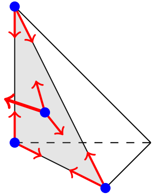

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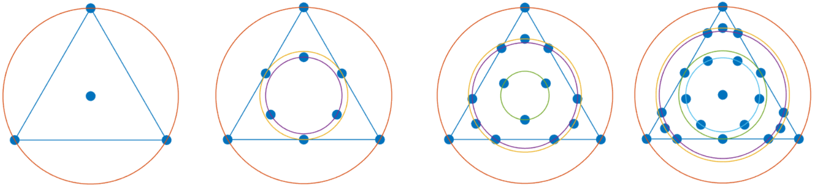
Note
Numerical experiments suggest the unmodified method yields second order convergence as well, but it does not fit our theory

Extension to even higher orders

We look for Gauss-Lobatto type quadrature rules !



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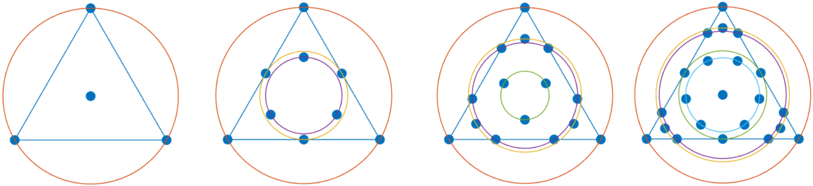

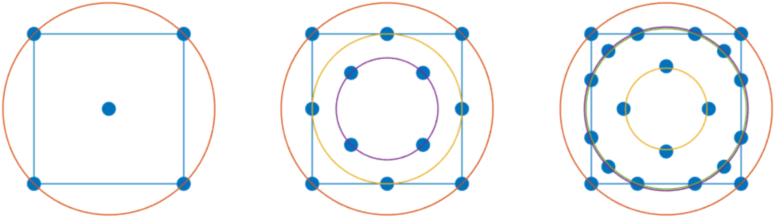

Closing remarks

- The discontinuous Galerkin method does outperform mass lumping for high orders.
  2018 - Geevers, Mulder, Vegt - New higher-order mass-lumped tetrahedral elements for wave propagation modelling
- Easier to define quadrature formulas on quadrilateral meshes because of tensor product structure (still not optimal!)
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Short recap

- We introduced several mass-lumped mixed finite element approximations for Maxwell's equations (lowest order)
- Developed weaker conditions on the exactness of the quadrature formula and extended the method to second order approximations