A mixed finite element method with mass lumping for wave propagation

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Motivation and existing methods

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\begin{aligned}
\partial_t u + \nabla p &= 0 && \text{in } \Omega \times (0, T), \\
\partial_t p + \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \\
p &= 0 && \text{on } \partial\Omega \times (0, T).
\end{aligned}
$$

with $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$.

Motivation and existing methods

Finite differences try [Yee 66]

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\partial_t u_1 + \partial_x p &= 0, \\
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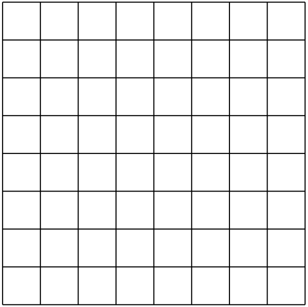

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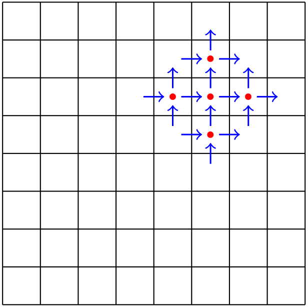

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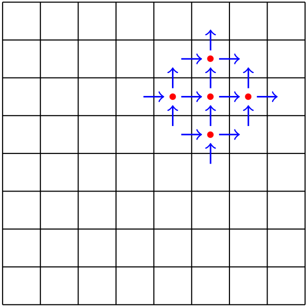

Pros

- Easy to implement
- Fast
- Optimal convergence

Cons

- Difficulties in dealing with complex domains

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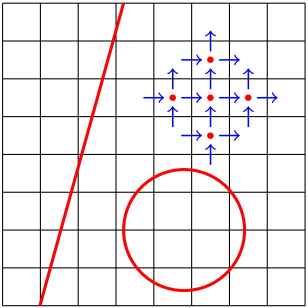

Pros

- Easy to implement
- Fast
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Motivation and existing methods

Important observation : For a structured mesh, the finite difference method is equivalent to a *modified* mixed finite element method [Cohen, Monk 97].

Can we develop a similar method on triangular meshes
that is *fast* and has *optimal convergence* ?

A new method

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\end{aligned}
$$

Variational characterization

Find $u(t) \in H(\operatorname{div}, \Omega)$ and $p(t) \in L^2(\Omega)$ such that

$$
\begin{aligned}
(\partial_t u(t), v) - (p(t), \operatorname{div} v) &= 0 \quad \forall v \in H(\operatorname{div}, \Omega),\\
(\partial_t p(t), q) + (\operatorname{div} u(t), q) &= 0 \quad \forall q \in L^2(\Omega).
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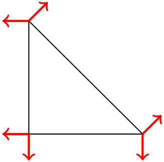
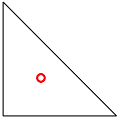
Galerkin discretization

Find $u_h(t) \in V_h \subseteq H(\operatorname{div}, \Omega)$ and $p_h(t) \in Q_h \subseteq L^2(\Omega)$ such that

$$
\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \operatorname{div} v_h) &= 0 \quad \forall v_h \in V_h,\\
(\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0 \quad \forall q_h \in Q_h.
\end{aligned}
$$

$$V_h = \mathrm{BDM}_1 := \mathrm{P}_1^2(\mathcal{T}_h) \cap H(\operatorname{div}, \Omega) \qquad Q_h = \mathrm{P}_0 := \mathrm{P}_0(\mathcal{T}_h)$$





A new method

$$(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
$$(\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) = 0 \qquad \forall q_h \in Q_h.$$

Algebraic system

$$M\,\dot{\mathbf{u}}_h - B^\top \mathbf{p}_h = 0$$
$$D\,\dot{\mathbf{p}}_h + B\,\mathbf{u}_h = 0$$

Problem : Structure of matrix M

A new method

$$
\begin{aligned}
(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0 \qquad \forall v_h \in V_h,\\
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Algebraic system

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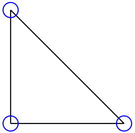
M_h \dot{\mathbf{u}}_h - B^\top \mathbf{p}_h &= 0\\
D \dot{\mathbf{p}}_h + B \mathbf{u}_h &= 0
\end{aligned}
$$

where $(\cdot,\cdot)_h$ is defined locally by

$$
(u_h, v_h)_{h,T} := \frac{|T|}{3} \sum_{i=1}^{3} u_h(x_i) v_h(x_i)
$$

where $\{x_i\}_{i=1,2,3}$ represent the vertices of the element.



This procedure is called MASS LUMPING !

A new method

$$(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) = 0 \qquad \forall v_h \in V_h,$$
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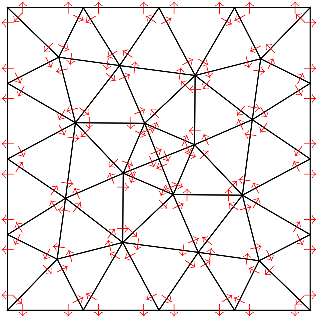
Representation of the degrees of freedom (left), structure of M^{-1} (right)



A new method

$$(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) = 0 \qquad \forall v_h \in V_h,$$
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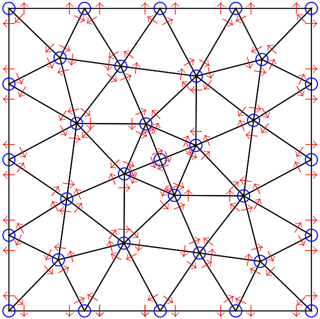

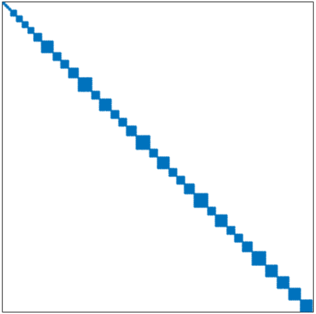

Analysis of the new method

Galerkin discretization

Find $u_h(t) \in V_h \subseteq H(\mathrm{div}\,, \Omega)$ and $p_h(t) \in Q_h \subseteq L^2(\Omega)$ such that

$$\begin{aligned}
(\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) &= 0 \quad \forall v_h \in V_h, \\
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\end{aligned}$$

Semi-discrete error

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} = O(h^2) \qquad \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} = O(h^2)$$

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Mass lumping kills second order convergence !

Analysis of the new method

Definition of a projection

Let the projection $\Pi_h u(t) = \Pi_h u(0) + \int_0^t \Pi_h \partial_t u(s)\, ds$ be defined via

$$(\Pi_h u(0), v_h)_h - (r_h(0), \mathrm{div}\ v_h) = (u(0), v_h) \qquad \forall v_h \in V_h,$$
$$(\mathrm{div}\ \Pi_h u(0), q_h) = (\mathrm{div}\ u(0), q_h) \qquad \forall q_h \in Q_h,$$

and
$$(\Pi_h \partial_t u(t), v_h)_h - (r_h(t), \mathrm{div}\ v_h) = 0 \qquad \forall v_h \in V_h,$$
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Discrete error

Let Ω be convex. For $p_h(0) = \pi_h^0 p(0)$ and $u_h(0) = \Pi_h u(0)$, we have

$$\|\Pi_h u(t) - u_h(t)\|_{L^2(\Omega)} = O(h^2) \qquad \|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} = O(h^2)$$

Proof : use elliptic result of [Wheeler, Yotov 06]

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Proof : use elliptic result of [Wheeler, Yotov 06]

The method contains second order information !

Post-processing strategies for the new method

Problem

For all $K \in \mathcal{T}_h$, $t > 0$ find $\widetilde{p}_h(t) \in P_1(K)$ such that

$$
\begin{aligned}
(\nabla \widetilde{p}_h(t), \nabla \widetilde{q}_h)_K &= -(\partial_t u_h(t), \nabla \widetilde{q}_h)_K \quad && \forall \widetilde{q}_h \in P_1(K), \\
(\widetilde{p}_h(t), q_h)_K &= (p_h(t), q_h)_K && \forall q_h \in P_0(K).
\end{aligned}
$$

Post-processing strategies for the new method

Problem

For all $K \in \mathcal{T}_h$, $t > 0$ find $\widetilde{p}_h(t) \in P_1(K)$ such that

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$$(\widetilde{p}_h(t), q_h)_K = (p_h(t), q_h)_K \qquad \forall q_h \in P_0(K).$$

Theorem

Let Ω be convex. Then

$$\|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} = O(h^2).$$

Idea from [Stenberg, 91]. Requirements for the proof :

$$\|\pi_h^0 p(t) - p_h(t)\|_{L^2(\Omega)} = O(h^2).$$

Only $\quad \|\partial_t u(t) - \partial_t u_h(t)\|_{L^2(\Omega)} = O(h).$

Post-processing strategies for the new method

Post-processing strategy for the velocity

For every $0 \leq t \leq T$, find $\widetilde{u}_h(t) \in V_h$ such that

$$
\begin{aligned}
(\widetilde{u}_h(t), v_h) - (\widetilde{r}_h(t), \mathrm{div}\ v_h) &= (u_h(t), v_h)_h \qquad && \forall v_h \in V_h, \\
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Error estimate for the improved velocity

Let Ω be convex. For $p_h(0) = \pi_h^0 p(0)$ and $u_h(0) = \Pi_h u(0)$ we have

$$\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} = O(h^2)$$

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Local in time, but global in space !

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\end{aligned}
$$

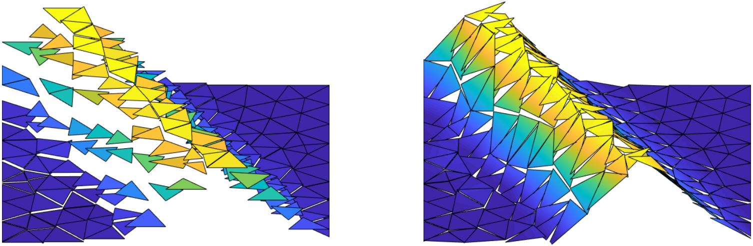
Full post-processing error

Let Ω be convex. For $p_h(0) = \pi_h^0 p(0)$ and $u_h(0) = \Pi_h u(0)$ we have

$$
\|u(t) - \widetilde{u}_h(t)\|_{L^2(\Omega)} + \|p(t) - \widetilde{p}_h(t)\|_{L^2(\Omega)} = O(h^2).
$$

Numerical examples

We highlight the qualitative improvement obtained by post-processing by means of the plane wave



Pressure p_h (left), post-processed pressure $\widetilde{p}_h$ (right)

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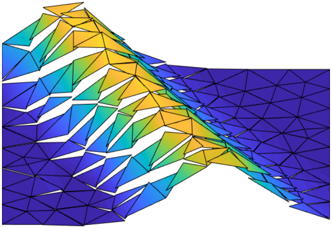
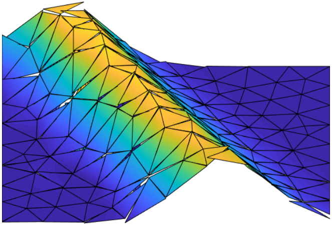
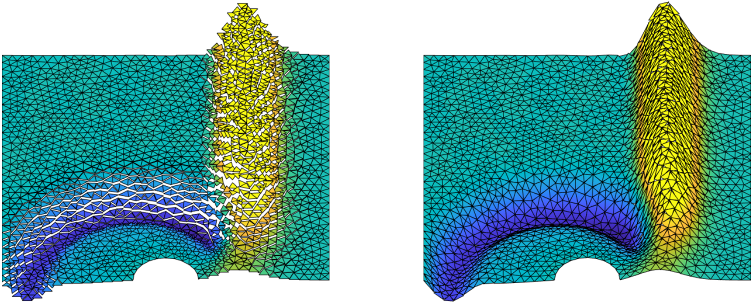




First component of the velocity u_h (left), first component of the post-processed velocity $\widetilde{u}_h$ (right)

Numerical examples

Convexity does not seem to be a necessary condition ...



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Summary

- We developed a mixed finite element method with mass lumping.
- We showed that the discrete solutions exhibit superconvergence with respect to carefully defined projections of the true solutions.
- We proposed post-processing strategies for both variables.

For further details, refer to [Egger, Radu 18, arXiv:1803.04238]

Remarks

- Extension to the a fully discrete scheme by the explicit leapfrog scheme.
- Only for lowest order $V_h = \mathrm{BDM}_1$ and $Q_h = \mathrm{P}_0$.
- The convexity condition could be further relaxed.

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Thank you for your attention

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