# A second order multipoint flux mixed finite element method on hybrid meshes

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# Porous media modeling

Model equations for single-phase flow:

| Conservation of mass | Darcy's law |
| --- | --- |
| $\operatorname{div} \mathbf{u} = f$ | $\mathbf{u} = -K \nabla p$ |

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Quantity of interest : $p$

Second order form

$$-\operatorname{div}(K\nabla p) = f \qquad \text{in } \Omega$$
$$p = 0 \qquad \text{on } \partial\Omega.$$

$(i)$ Discontinuous schemes (DFVM), (DG) for local mass conservation

$(ii)$ Not accurate for rough coefficients (local arithmetic averaging of $K$)

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($i$) Discontinuous schemes (DFVM), (DG) for local mass conservation

($ii$) Not accurate for rough coefficients (local arithmetic averaging of $K$)

**Mixed form**

$$\begin{aligned} K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

($i$) Handles rough coefficients better (local harmonic averaging of $K$)

($ii$) Have to solve a full saddle point problem... or do you ? $\Rightarrow$ MFMFE

# Variational formulation

$$\begin{aligned} K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Variational formulation

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= 0 && \forall \mathbf{v} \in H(\operatorname{div}, \Omega) \\ (\operatorname{div} \mathbf{u}, q) &= (f, q) && \forall q \in L^2(\Omega) \end{aligned}$$

# Discrete variational formulation

$$
\begin{aligned}
K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\
\operatorname{div} \mathbf{u} &= f && \text{in } \Omega \\
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\end{aligned}
$$

Discrete variational formulation

$$
\begin{aligned}
(K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) &= 0 && \forall \mathbf{v}_h \in \mathbf{V}_h \subseteq H(\operatorname{div}, \Omega) \\
(\operatorname{div} \mathbf{u}_h, q_h) &= (f, q_h) && \forall q_h \in Q_h \subseteq L^2(\Omega)
\end{aligned}
$$

Problem : we have to solve a full (indefinite) saddle point system ...

# Mass lumping

$$\begin{aligned} K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Discrete variational formulation via *mass lumping* (MFMFE)

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h - (p_h, \operatorname{div} \mathbf{v}_h) &= 0 && \forall \mathbf{v}_h \in \mathbf{V}_h \subseteq H(\operatorname{div}, \Omega) \\ (\operatorname{div} \mathbf{u}_h, q_h) &= (f, q_h) && \forall q_h \in Q_h \subseteq L^2(\Omega) \end{aligned}$$

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For appropriate spaces $\mathbf{V}_h$, $Q_h$ and $(\cdot,\cdot)_h$, the *lumped mass matrix* $M_h$ is block-diagonal, and the variable $\mathbf{u}_h$ can be eliminated efficiently.

$$\begin{pmatrix} \mathsf{M_h} & -\mathsf{C}^\top \\ \mathsf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u} \\ \mathsf{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f} \end{pmatrix} \qquad \Longrightarrow \qquad \mathsf{C}\mathsf{M_h}^{-1}\mathsf{C}^\top \mathsf{p} = \mathsf{f}$$

The problem reduces to symmetric, positive definite cell-centered system for the pressure (CCFD)

# Discretization

Discrete variational formulation via *mass lumping* (MFMFE)

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M. Wheeler, I. Yotov *A multipoint flux mixed finite element method*. SIAM 2006

$V(T) = \mathsf{BDM}_1(T) := P_1(T)^2$ $\qquad$ $(\mathbf{u}_h, \mathbf{v}_h)_h := \frac{|T|}{3} \sum_{i=1}^{3} \mathbf{u}_h(r_i)\mathbf{v}_h(r_i)$

$Q(T) = P_0(T)$ $\qquad$ $r_i$ vertex

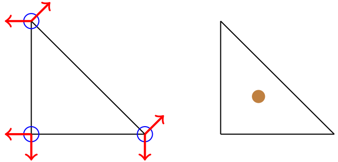

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points.

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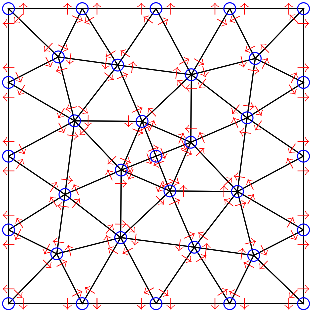

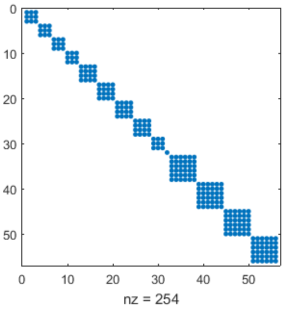

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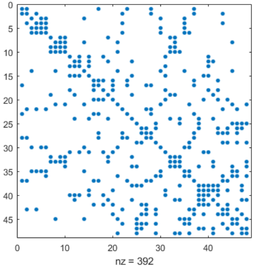


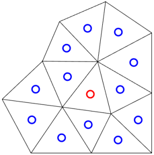

Figure: Matrix $CM_h^{-1}C^T$ (left), stencil of the method (right)

# Convergence analysis

Summary of the convergence results

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h) \qquad \text{and} \qquad \|\pi_h^0 p - p_h\| = O(h^2)$$

Relevant properties

- $(i)$ $P_0(T)^2 \subseteq \mathbf{V}(T)$ and $P_0(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$
- $(ii)$ The quadrature rule is exact for $P_0(T)^2 \times \mathbf{V}(T)$
- $(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$

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$(ii)$ The quadrature rule is exact for $P_0(T)^2 \times \mathbf{V}(T)$ ✓

$(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

**Wheeler-Yotov element** : $\mathbf{V}(T) = \mathrm{BDM}_1(T) = P_1(T)^2$

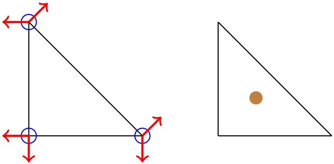

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_1(T)$.

# Higher order candidates

Natural extension of the first order estimates

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h^2) \qquad \text{and} \qquad \|\pi_h^1 p - p_h\| = O(h^3)$$

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$(i)$ $P_1(T)^2 \subseteq \mathbf{V}(T)$ and $P_1(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$

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Relevant properties

$(i)$ $P_1(T)^2 \subseteq \mathbf{V}(T)$ and $P_1(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$ ✓

$(ii)$ The quadrature rule is exact for $P_1(T)^2 \times \mathbf{V}(T)$ ✗

$(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$ ✗

**First candidate** : $\mathbf{V}(T) = \mathrm{BDM}_2(T) = P_2(T)^2$

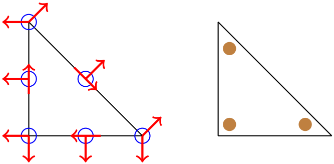

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points.

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Natural extension of the first order estimates

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h^2) \qquad \text{and} \qquad \|\pi_h^1 p - p_h\| = O(h^3)$$

Relevant properties

$(i)$ $P_1(T)^2 \subseteq \mathbf{V}(T)$ and $P_1(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$ ✓

$(ii)$ The quadrature rule is exact for $P_1(T)^2 \times \mathbf{V}(T)$ ✓

$(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

**Second candidate** : $\mathbf{V}(T) = \mathrm{BDM}_2^+(T) = P_2(T)^2 \oplus b_3 \cdot [1,0]^T \oplus b_3 \cdot [0,1]^T$

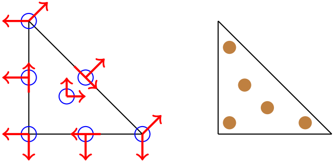

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_3(T) \oplus b_3 \cdot P_1(T)$.

# Higher order candidates

Natural extension of the first order estimates

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h^2) \qquad \text{and} \qquad \|\pi_h^1 p - p_h\| = O(h^3)$$

Relevant properties

$(i)$ $P_1(T)^2 \subseteq \mathbf{V}(T)$ and $P_1(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$ ✓

$(ii)$ The quadrature rule is exact for $P_1(T)^2 \times \mathbf{V}(T)$ ✗

$(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

**Third candidate** : $\mathbf{V}(T) = \mathrm{RT}_1(T) := P_1(T)^2 + \mathbf{x} \cdot P_1^h(T)$

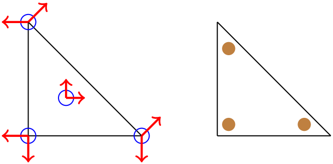

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_2(T)$.

# A new theory

Split the error in $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \|\mathbf{u} - \Pi_h \mathbf{u}\|_{L^2(\Omega)} + \|\Pi_h \mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$

$$(\Pi_h \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)_h - (\pi_h^1 p - p_h, \operatorname{div} \mathbf{v}_h) = (\Pi_h \mathbf{u} - \mathbf{u}, \mathbf{v}_h) + \sigma_h(\Pi_h \mathbf{u}, v_h)$$
$$(\operatorname{div}(\Pi_h \mathbf{u} - \mathbf{u}_h), q_h) = 0$$

with $\sigma_h(\Pi_h \mathbf{u}, \mathbf{v}_h) = (\Pi_h \mathbf{u}, \mathbf{v}_h)_h - (\Pi_h \mathbf{u}, \mathbf{v}_h)$.

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$$(\Pi_h\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h)_h - (\pi_h^1 p - p_h, \operatorname{div}\mathbf{v}_h) = (\Pi_h\mathbf{u}-\mathbf{u}, \mathbf{v}_h) + \sigma_h(\Pi_h\mathbf{u}, v_h)$$
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**(I)** $\operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h) = 0 \quad \Rightarrow \quad \Pi_h\mathbf{u}-\mathbf{u}_h \in P_1(T)^2$

**(II)** $\sigma_h(\mathbf{u}_h, \mathbf{v}_h) = 0$ if $\mathbf{u}_h, \mathbf{v}_h \in P_1(T)^2$

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$$
\begin{aligned}
(\Pi_h\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)_h - (\pi_h^1 p - p_h, \operatorname{div}\mathbf{v}_h) &= (\Pi_h\mathbf{u}-\mathbf{u},\mathbf{v}_h) + \sigma_h(\Pi_h\mathbf{u}, v_h) \\
(\operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h), q_h) &= 0
\end{aligned}
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with $\sigma_h(\Pi_h\mathbf{u},\mathbf{v}_h) = (\Pi_h\mathbf{u},\mathbf{v}_h)_h - (\Pi_h\mathbf{u},\mathbf{v}_h)$.

(I) $\operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h) = 0 \quad \Rightarrow \quad \Pi_h\mathbf{u}-\mathbf{u}_h \in P_1(T)^2$

(II) $\sigma_h(\mathbf{u}_h,\mathbf{v}_h) = 0$ if $\mathbf{u}_h,\mathbf{v}_h \in P_1(T)^2$

Taking $\mathbf{v}_h = \Pi_h\mathbf{u}-\mathbf{u}_h$ and $q_h = \pi_h^1 p - p_h$, we obtain

$$
\begin{aligned}
\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_h^2 &= (\Pi_h\mathbf{u}-\mathbf{u},\Pi_h\mathbf{u}-\mathbf{u}_h) + \sigma_h(\Pi_h\mathbf{u},\Pi_h\mathbf{u}-\mathbf{u}_h) \\
&= (\Pi_h\mathbf{u}-\mathbf{u},\Pi_h\mathbf{u}-\mathbf{u}_h) + \sigma_h(\Pi_h\mathbf{u}-\pi_h^1\mathbf{u},\Pi_h\mathbf{u}-\mathbf{u}_h) \\
&\leq \|\Pi_h\mathbf{u}-\mathbf{u}\|_0\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0 + c\|\Pi_h\mathbf{u}-\pi_h^1\mathbf{u}\|_0\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0 \\
&\leq Ch^2\|\mathbf{u}\|_{H^2(\mathcal{T}_h)}\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0
\end{aligned}
$$

# A new theory

**Theorem**

$$\|\mathbf{u}-\mathbf{u}_h\| = O(h^2) \qquad \textit{and} \qquad \|\pi_h^0(p-p_h)\| = O(h^3)$$

Relevant properties

$(i)$ $P_1(T)^2 \subset \mathbf{V}(T)$ and $P_1(T) \subset Q(T)$ such that $\operatorname{div}\mathbf{V}(T) \subseteq Q(T)$ ✓

$(ii_a)$ $\exists\, \widetilde{\mathbf{V}}(T) \subset \mathbf{V}(T)$ s.t. $\mathbf{v} \in \mathbf{V}(T)$ with $\operatorname{div}\mathbf{v} \in \operatorname{div}\widetilde{\mathbf{V}}(T)$ imply $\mathbf{v} \in \widetilde{\mathbf{V}}(T)$ ✓

$(ii_b)$ The quadrature rule is exact for $P_1(T)^2 \times \widetilde{\mathbf{V}}(T)$ ✓

$(iii)$ The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

**Third candidate** : $\mathbf{V}(T) = \mathrm{RT}_1(T) := P_1(T)^2 + \mathbf{x}\cdot P_1^h(T)$

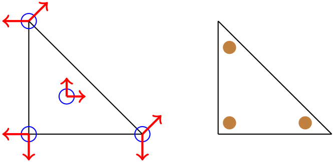


Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points.

# Numerical tests

$$p = \sin(\pi x)\sin(\pi y) \qquad K = \begin{pmatrix} 4 + (x+2)^2 + y^2 & 1 + \sin(xy) \\ 1 + \sin(xy) & 2 \end{pmatrix}$$

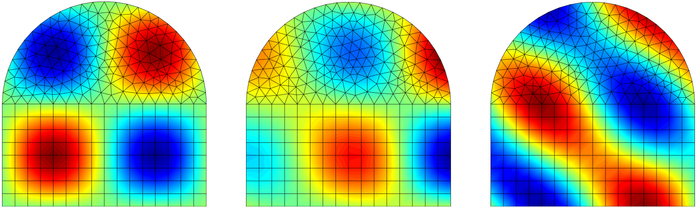
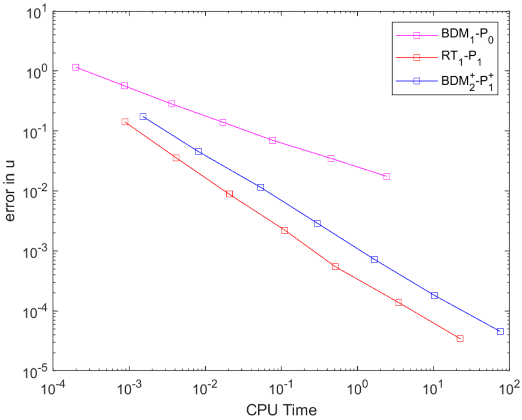


Figure: Snapshots of the pressure $p_h$ (left) and the two velocity components $u_{x,h}$, $u_{y,h}$ (middle, right) for the second order approximation.

| $h$ | DOF $u$ | DOF $p$ | $\|u - u_h\|$ | eoc | $\|\pi_h^0(p - p_h)\|$ | eoc |
|---|---|---|---|---|---|---|
| $2^{-1}$ | 164 | 84 | 0.078309 | — | 0.033106 | — |
| $2^{-2}$ | 724 | 396 | 0.013097 | 2.57 | 0.002864 | 3.53 |
| $2^{-3}$ | 2498 | 1386 | 0.002275 | 2.52 | 0.000391 | 2.87 |
| $2^{-4}$ | 9738 | 5466 | 0.000484 | 2.23 | 0.000049 | 2.99 |
| $2^{-5}$ | 40230 | 22770 | 0.000099 | 2.28 | 0.000005 | 3.13 |

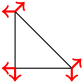
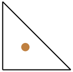
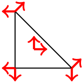
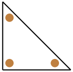
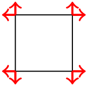
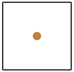
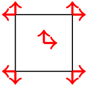
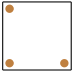
Table: Degrees of freedom, relative discretization errors, and convergence rates for the second order multipoint flux finite element method.

# Comparison




The $\mathrm{RT}_1 - \mathrm{P}_1$ pair is about 3x faster than the $\mathrm{BDM}_2^+ - \mathrm{P}_1^+$ pair.

# Hybrid meshes

| | dim $\mathbf{V}(T)$ | dim $Q(T)$ | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\|\pi_h^0 p - p_h\|_0$ | DOFs for $\mathbf{u}_h$ | DOFs for $p_h$ |
|---|---|---|---|---|---|---|
| $\mathrm{BDM}_1 - \mathrm{P}_0$ | 6+0 | 1 | $O(h)$ | $O(h^2)$ |  |  |
| $\mathrm{RT}_1 - \mathrm{P}_1$ | 6+2 | 3 | $O(h^2)$ | $O(h^3)$ |  |  |
| $\mathrm{BDM}_1 - \mathrm{P}_0$ | 8+0 | 1 | $O(h)$ | $O(h^2)$ |  |  |
| $\mathrm{BDFM}_2 - \mathrm{P}_1$ | 8+2 | 3 | $O(h^2)$ | $O(h^3)$ |  |  |

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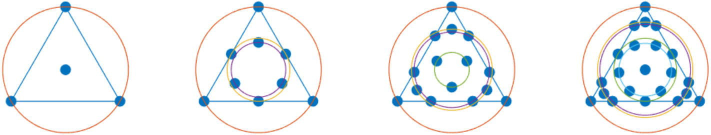

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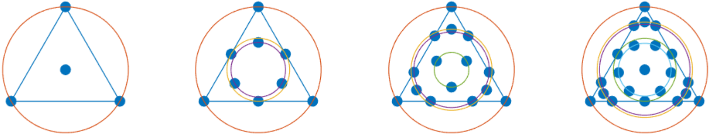

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(iv) Application to wave propagation

$$\begin{aligned} \partial_t \mathbf{u} + \nabla p &= f && \text{in } \Omega \\ \partial_t p + \operatorname{div} \mathbf{u} &= g && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

# Extension to poroelasicity

Poroelastic consolidation (three-field formulation)

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K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\
c_s \partial_t p + \operatorname{div}(\alpha \partial_t \mathbf{w}) + \operatorname{div} \mathbf{u} &= f && \text{in } \Omega \\
-\operatorname{div}(2\mu\, \varepsilon(\mathbf{w}) + \lambda \operatorname{div}(\mathbf{w}) I) + \alpha \nabla p &= g && \text{in } \Omega
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\mathsf{A}\,\mathsf{w} - \mathsf{B}^\top \mathsf{p} \qquad\qquad\qquad &= \mathsf{g}
\end{aligned}
$$

(iii) Algebraically, we solve the two-field formulation for $p$ and $\mathbf{w}$, which is more efficient.

# Summary

- ➔ Introduced the multipoint flux mixed finite element method (MFMFE)
- ➔ Presented the first order approximation introduced by Wheeler and Yotov
- ➔ Proposed an extension to second order approximations

H. Egger and B. Radu.
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A few additional remarks

- ➔ Extension to the $3D$ case has also been done.
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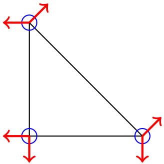
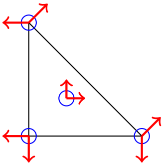
## Thank you for your attention

Acknowledgement : The work of Bogdan Radu is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universität Darmstadt

## Triangles

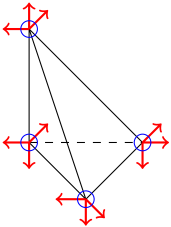


(a) $BDM_1$ first order element



(b) $RT_1$ second order element

## Tetrahedra



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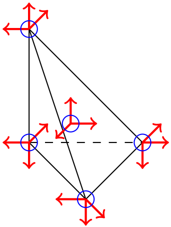
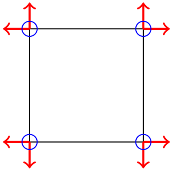


(b) $RT_1$ second order element

## Parallelograms



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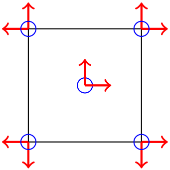
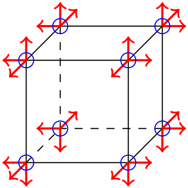
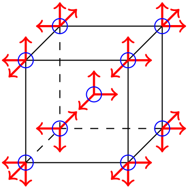


(b) $BDFM_2$ second order element

## Hexahedra



(a) $eBDDF_1$ first order element



(b) ? - second order element