A second order finite element method with mass lumping for Maxwell's equations on tetrahedra

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Maxwell's equations

Electromagnetic wave propagation in linear and non-dispersive but possibly inhomogeneous and anisotropic media

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in Ω, with $E(0) = E_0$ and $H(0) = H_0$ in Ω and $n \times E(t) = 0$ on $\partial\Omega$

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Methods: FDTD/FIT, FEM, FVM, DG, ...

Finite differences (FDTD/FIT)

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- 1966 - Yee - Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media
- 1977 - Weiland - Eine Methode zur Lösung der Maxwell'schen Gleichungen für sechskomponentige Felder auf diskreter Basis
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Pros

- Easy to implement
- stable, accurate $O(h^2 + \tau^2)$, efficient

Cons

- Difficulties in dealing with complex domains

Finite element method

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Approximation spaces: $V_h \subset H_0(\operatorname{curl}, \Omega)$ and $Q_h \subset L^2(\Omega)$

Galerkin method: For $t > 0$, find $E_h(t) \in V_h$ and $H_h(t) \in Q_h$ such that

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Algebraic realization. For a choice of basis functions, we have

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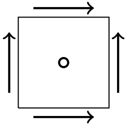
First order elements

Finite element spaces on reference elements.

- 1980 - Nedelec - Mixed Finite Elements in $\mathbb{R}^3$



$$V_h(Q) = \mathcal{N}_0(Q) \qquad \phi_1 = (1-y, 0) \qquad \phi_3 = (0, 1-x)$$
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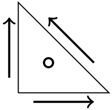

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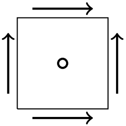

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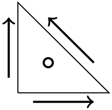

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Lemma (accuracy) If E and H are sufficiently smooth, then

$$\|E(t) - E_h(t)\|_{L^2} + \|H(t) - H_h(t)\|_{L^2} \leq Ch$$

- 1992 - Monk - Analysis of a finite element method for Maxwell's equations
- 1993 - Monk - An analysis of Nedelec's method for spatial discretization of Maxwell's equations

First order elements

Stability and accuracy.
Lowest order MFEM yields stable and accurate approximation in space.

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Numerical solution. Time integration of resulting ode system

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by explicit schemes requires application of $\mathbf{M}_\varepsilon^{-1}$ and $\mathbf{D}_\mu^{-1}$.

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Thus, explicit time-stepping for standard MFEM is not efficient.

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Remedy – Mass-lumping: replace $\mathbf{M}_\varepsilon$ by approximation $\mathbf{M}_\varepsilon^L$ such that

- $\mathbf{M}_\varepsilon^L$ corresponds to positive definite matrix (stability)
- $\mathbf{M}_\varepsilon^L$ is good approximation for $\mathbf{M}_\varepsilon$ (accuracy)
- $(\mathbf{M}_\varepsilon^L)^{-1}$ can be applied efficiently (efficiency)

construction of $\mathbf{M}_\varepsilon^L$ usually via numerical quadrature.

Mass lumping literature

- 1975 - Fried, Malkus - Finite element mass matrix lumping by numerical integration with no convergence rate loss
- 1999 - Kong, Mulder, Veldhuizen - Higher-order triangular and tetrahedral finite elements with mass lumping for solving the wave equation
- 2000 - Becache, Joly, Tsogka - An analysis of new mixed finite elements for the approximation of wave propagation models
- 2001 - Mulder - Higher-order mass-lumped finite elements for the wave equation
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- 2018 - Geevers, Mulder, Vegt - New higher-order mass-lumped tetrahedral elements for wave propagation modelling

Mass-lumping in H^1

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- 1990 - Lee, Madsen - A mixed FEM formulation for Maxwell's equations in the time domain
- 1995 - Cohen, Monk - Mass lumped edge elements in three dimensions
- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 3D
- 1998 - Cohen, Monk - Gauss Point Mass Lumping Schemes for Maxwell's Equations
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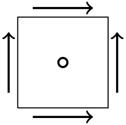
- 2004 - Lacoste - Mass-lumping for the first order Raviart–Thomas–Nedelec finite elements
- 2007 - Jund, Salmon - Arbitrary high-order finite element schemes and high order mass lumping
- 2018 - Geevers, Mulder, Vegt - New higher-order mass-lumped tetrahedral elements for wave propagation modelling

Mass-lumping in H^1

Mass-lumping in $H(\mathrm{div})$ and $H(\mathrm{curl})$

Observation for the lowest order case



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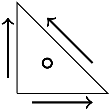

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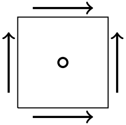

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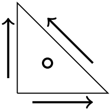

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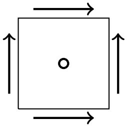

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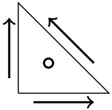

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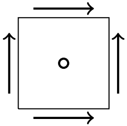

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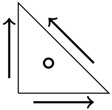

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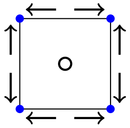
Some existing methods: Acute mesh lumping (triangles)

- 1996 - Baranger - Connection between finite volume and mixed finite element methods

First order elements... with mass lumping!

Use a larger polynomial space



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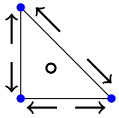

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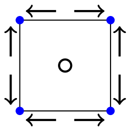

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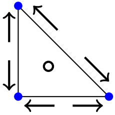

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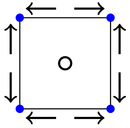

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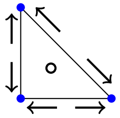

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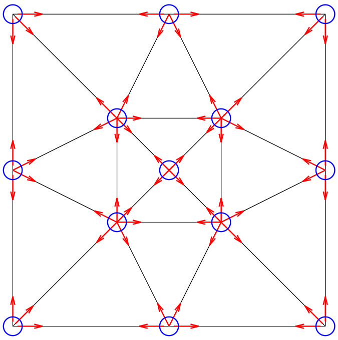

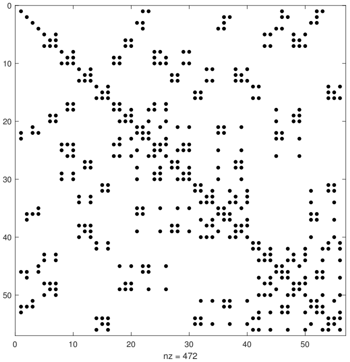

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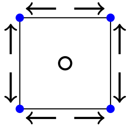

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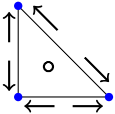

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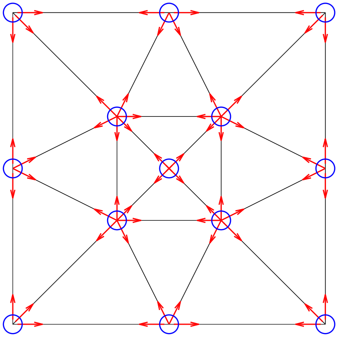

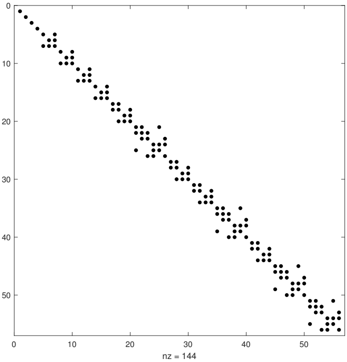

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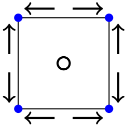

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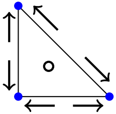

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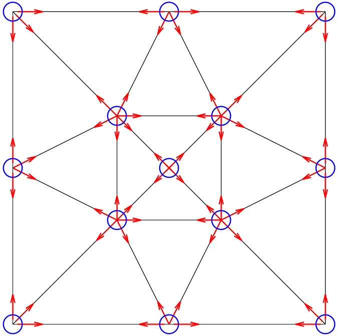

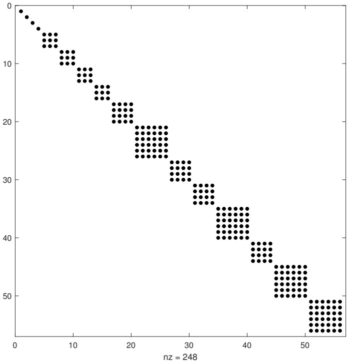

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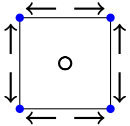

$\widetilde{V}_h(Q) = \mathcal{NC}_1(Q)$
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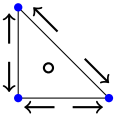

$\widetilde{V}_h(T) = \mathcal{NC}_1(T)$
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Lemma. $\widetilde{\mathbf{M}}_\epsilon^L$ is block diagonal and thus also $(\widetilde{\mathbf{M}}_\epsilon^L)^{-1}$.

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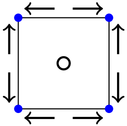
If $\boldsymbol{E}$ and $\boldsymbol{H}$ are sufficiently smooth, then

$$\|\boldsymbol{E}(t) - \widetilde{\boldsymbol{E}}_h(t)\| + \|\boldsymbol{H}(t) - \widetilde{\boldsymbol{H}}_h(t)\| \leq Ch$$

- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations

First order elements... with mass lumping!

Use a larger polynomial space



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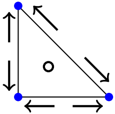

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Proof Idea: Error splitting in discrete and projection error, discrete stability, energy estimates, consistency error, analysis of the quadrature error (Strang).

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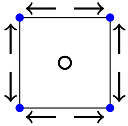

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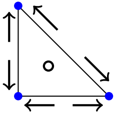

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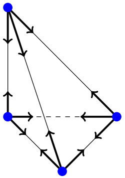
Requirement : The quadrature rule must be exact for $P_0(T)^2 \times \widetilde{V}_h(T)$

First order elements on tetrahedral meshes

The same concept also applies in 3D on tetrahedral meshes



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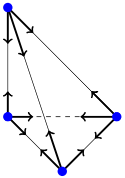

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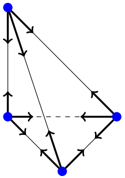

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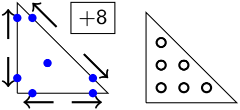
Next task : Extension to second order elements.

Second order elements

Extension to second order elements

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 2D



$$\widehat{V}_h(T) = \mathcal{N}_1(T) \oplus B = \mathcal{E}\mathcal{J}_1(T) \subseteq P_3(T)$$
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Second order elements

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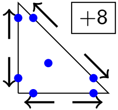

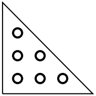

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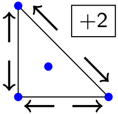
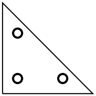
New proposal :





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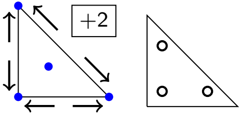
The quadrature rule is exact for P_2 polynomials ... but is this enough ?

Short notes on the analysis

New proposal :



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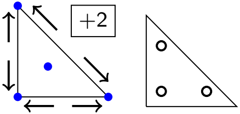

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Proof Idea: Discrete stability, energy estimates, Galerkin orthogonality, consistency error, Strang analysis of the quadrature error.

Classic requirement : The quadrature rule has to be exact for $P_1(T)^d \times \widehat{V}_h(T)$

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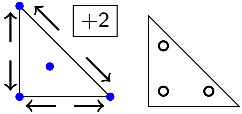

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New requirements

(i) There exists a splitting $\widehat{V}_h(T) = W(T) \oplus B(T)$ such that $\dim(B(T)) = \dim(\mathrm{curl}(B(T)))$ and $\mathrm{curl}(B(T)) \cap \mathrm{curl}(W(T)) = \{0\}$

(ii) The quadrature rule is exact for $P_1(T)^2 \times W(T)$

Short notes on the analysis

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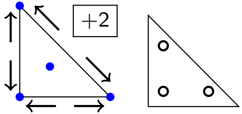

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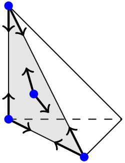
(i) There exists a splitting $\widehat{V}_h(T) = \mathcal{NC}_1(T) \oplus B(T)$ such that $\dim(B(T)) = \dim(\mathrm{curl}(B(T)))$ and $\mathrm{curl}(B(T)) \cap \mathrm{curl}(\mathcal{NC}_1(T)) = \{0\}$

(ii) The quadrature rule is exact for $P_1(T)^2 \times \mathcal{NC}_1(T) = P_2(T)^2$

Second order method - 3D

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 3D



$$\widehat{V}_h(T) = \mathcal{N}_1(T)$$

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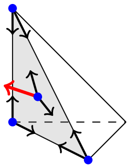

$$\widehat{V}_h(T) = \mathcal{N}_1(T) \oplus B(T) \qquad B(T) = \text{span} \left\{ \begin{array}{l} \Phi_1 = \lambda_2\lambda_3\lambda_4\nabla\lambda_1 \\ \Phi_2 = \lambda_1\lambda_3\lambda_4\nabla\lambda_2 \\ \Phi_3 = \lambda_1\lambda_2\lambda_4\nabla\lambda_3 \\ \Phi_4 = \lambda_1\lambda_2\lambda_3\nabla\lambda_4 \end{array} \right\}$$

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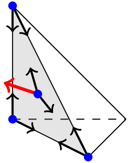

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But $\nabla(\lambda_1\lambda_2\lambda_3\lambda_4) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \Rightarrow \text{curl}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = 0.$

Thus $\dim(B(T)) \neq \dim(\text{curl}\, B(T))$!

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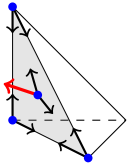

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Theorem [EggerRadu21]. If (and only if) $\text{div}(\boldsymbol{E}) = 0$, then

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Note. In general, second order convergence is lost!

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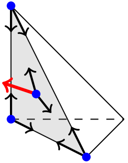

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Solution. Modify one basis function, for example $\widehat{\Phi}_4 = \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1 + 1)\nabla\lambda_4$

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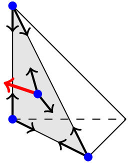

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Main takeaways

Key ingredients for mass lumping:

- Start with a basis space V_h that contains all $P_k(T)^d$ polynomials (for approximation).
- V_h dictates the number of continuity conditions on the boundary
- Find a quadrature rule that has sufficiently many quadrature points on the boundary and has the desired accuracy
- Extend V_h by appropriate "bubble" functions such that we have exactly d-many functions for each quadrature point.

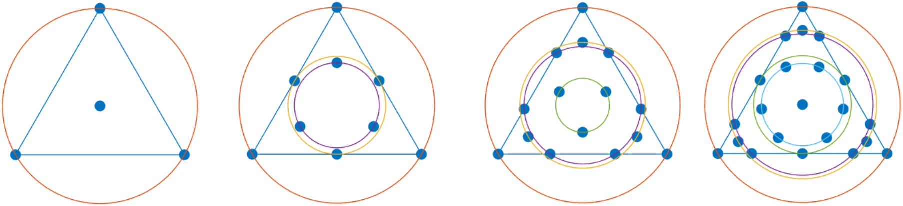
List of relevant publications

- 2020 - Egger, Radu - A mass-lumped mixed finite element method for acoustic wave propagation.
- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations
- 2021 - Egger, Radu - A second order finite element method with mass lumping for wave equations in H(div).
- 2021 - Egger, Radu - A Second-Order Finite Element Method with Mass Lumping for Maxwell's Equations on Tetrahedra.

Thank you for your attention!

Extension to even higher orders

We look for Gauss-Lobatto type quadrature rules !



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