Superconvergence and postprocessing for mixed finite element approximations of the wave equation

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Acoustic wave equation

$$
\begin{aligned}
\partial_t p + \operatorname{div} u &= f && \text{in } \Omega \times (0, T), \\
\partial_t u + \nabla p &= g && \text{in } \Omega \times (0, T), \\
p &= 0 && \text{on } \partial\Omega \times (0, T)
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\end{aligned}
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Remark (Existence and uniqueness)

Existence and uniqueness of a solution

$$(p, u) \in C([0, T], H_0^1(\Omega) \times H(\operatorname{div}, \Omega)) \cap C^1([0, T], L^2(\Omega) \times L^2(\Omega)^2)$$

for suitable initial and right hand side data follows from the semigroup theory.

Variational formulation

$$\begin{aligned} \partial_t p + \operatorname{div} u &= f && \text{in } \Omega \times (0,T), \\ \partial_t u + \nabla p &= g && \text{in } \Omega \times (0,T), \\ p &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Variational characterization

$$\begin{aligned} (\partial_t p(t), q) + (\operatorname{div} u(t), q) &= (f(t), q) \quad \forall q \in L^2(\Omega) \\ (\partial_t u(t), v) - (p(t), \operatorname{div} v) &= (g(t), v) \quad \forall v \in H(\operatorname{div}, \Omega) \end{aligned}$$

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Remark

Each classical solution satisfies the variational characterization.

Remark

The spaces corresponding to the weak formulation are $L^2(\Omega)$ for p and $H(\operatorname{div}, \Omega)$ for u.

Semi-discretization

Problem

For $(p_h(0), u_h(0)) = (\pi_{L^2} p_0, \rho_h u_0)$ *and all* $t > 0$ *find* $(p_h(t), u_h(t)) \in Q_h \times V_h$ *such that*

$$\begin{aligned} (\partial_t p_h(t), q_h) + (\mathrm{div}\, u_h(t), q_h) &= (f(t), q_h) \qquad \forall q_h \in Q_h \\ (\partial_t u_h(t), v_h) - (p_h(t), \mathrm{div}\, v_h) &= (g(t), v_h) \qquad \forall v_h \in V_h \end{aligned}$$

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Lemma (Discrete energy estimate)

Existence and uniqueness granted by Picard-Lindelöf. Moreover, we have

$$\begin{aligned} &\|p_h(t)\|_{L^2} + \|u_h(t)\|_{L^2} \leq \\ &\qquad \leq C \left(\|p_h(0)\|_{L^2} + \|u_h(0)\|_{L^2} + \int_0^t (\|f(s)\|_{L^2} + \|g(s)\|_{L^2})\, ds \right). \end{aligned}$$

Discrete spaces

$$Q_h = \mathsf{P}_0 := \mathsf{P}_0(\mathcal{T}_h) \qquad V_h = \mathsf{BDM}_1 := \mathsf{P}_1^2(\mathcal{T}_h) \cap H(\operatorname{div}, \Omega) \qquad \operatorname{div} V_h = Q_h$$

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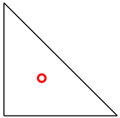

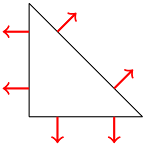

Discrete spaces

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Projection operators $\pi_{L^2} : L^2(\Omega) \to Q_h$, $\rho_h : H^1(\mathcal{T}_h) \cap H(\mathrm{div}\,, \Omega) \to V_h$

$$\mathrm{div}\, \rho_h v = \pi_{L^2} \mathrm{div}\, v, \quad \forall v$$

$$\|p - \pi_{L^2} p\|_{L^2(\Omega)} \leq Ch|p|_{1,\Omega}$$

$$\|u - \rho_h u\|_{L^2(\Omega)} \leq Ch^2 |u|_{2,\Omega}.$$

Error estimate

Remark

Jenkins/Wheeler, Chen :

$$\|p(t) - p_h(t)\|_{L^2} + \|u(t) - u_h(t)\|_{L^2} \leq Ch.$$

- T. Geveci *On the application of mixed finite element methods to the wave equations*. RAIRO Model. Math. Anal. Numer. 1988
- E. W. Jenkins and T. Rivière and M. F. Wheeler *A priori error estimates for mixed finite element approximations of the acoustic wave equation*. SIAM J. Numer. Anal. 2002
- L. C. Cowsar and T. F. Dupont and M. F. Wheeler *A priori estimates for mixed finite element approximations of second-order hyperbolic equations with absorbing boundary conditions*. SIAM J. Numer. Anal. 1996

Error estimate

Theorem

Let $Q_h = P_0$, $V_h = BDM_1$. *Then*

$$\|\pi_{L^2} p(t) - p_h(t)\|_{L^2} + \|u(t) - u_h(t)\|_{L^2} \leq Ch^2 \|\partial_t u\|_{H^2(\Omega)}.$$

Error estimate

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$$\begin{aligned}\|p - p_h\|_{L^2} + \|u - u_h\|_{L^2} \leq \|p - \pi_{L^2} p\|_{L^2} + \|u - \rho_h u\|_{L^2} + \\ + \|\pi_{L^2} p - p_h\|_{L^2} + \|\rho_h u - u_h\|_{L^2}\end{aligned}$$

Error estimate

Theorem

Let $Q_h = P_0$, $V_h = BDM_1$. Then

$$\|\pi_{L^2} p(t) - p_h(t)\|_{L^2} + \|u(t) - u_h(t)\|_{L^2} \leq Ch^2 \|\partial_t u\|_{H^2(\Omega)}.$$

$$\|p - p_h\|_{L^2} + \|u - u_h\|_{L^2} \leq \overbrace{\|p - \pi_{L^2} p\|_{L^2}}^{O(h)} + \overbrace{\|u - \rho_h u\|_{L^2}}^{O(h^2)} + \\ + \|\pi_{L^2} p - p_h\|_{L^2} + \|\rho_h u - u_h\|_{L^2}$$

For $r_h = \pi_{L^2} p - p_h$ and $w_h = \rho_h u - u_h$, we have

$$(\partial_t r_h, q_h) + (\operatorname{div} w_h, q_h) = (\widetilde{f}(t), q_h)$$
$$(\partial_t w_h, v_h) - (r_h, \operatorname{div} v_h) = (\widetilde{g}(t), v_h)$$

with initial values $r_h(0) = 0$, $w_h(0) = 0$ and right hand sides

$$(\widetilde{f}(t), q_h) = (\partial_t(\pi_{L^2} p - p), q_h) + (\operatorname{div} \rho_h u - \operatorname{div} u), q_h) = 0$$
$$(\widetilde{g}(t), v_h) = (\partial_t(\rho_h u - u), v_h) - (\pi_{L^2} p - p, \operatorname{div} v_h) = (\partial_t(\rho_h u - u), v_h)$$

Error estimate

Theorem

Let $Q_h = P_0$, $V_h = BDM_1$. Then

$$\|\pi_{L^2} p(t) - p_h(t)\|_{L^2} + \|u(t) - u_h(t)\|_{L^2} \le Ch^2 \|\partial_t u\|_{H^2(\Omega)}.$$

$$\begin{aligned}\|p - p_h\|_{L^2} + \|u - u_h\|_{L^2} \le & \overbrace{\|p - \pi_{L^2} p\|_{L^2}}^{O(h)} + \overbrace{\|u - \rho_h u\|_{L^2}}^{O(h^2)} + \\ & + \underbrace{\|\pi_{L^2} p - p_h\|_{L^2} + \|\rho_h u - u_h\|_{L^2}}_{O(h^2)}\end{aligned}$$

Theorem

Let $Q_h = P_0$, $V_h = BDM_1$. Then

$$\|\partial_t(\pi_{L^2} p(t) - p_h(t))\|_{L^2} + \|\partial_t(u(t) - u_h(t))\|_{L^2} \le Ch^2 \|\partial_{tt} u\|_{H^2(\Omega)}.$$

Post-processing

Idea : Construct $\widetilde{p}_h \in P_1(\mathcal{T}_h)$ from p_h, u_h. Testing the momentum equation with $\nabla q \in L^2(\Omega)^2$ gives

$$(\nabla p, \nabla q)_K = -(\partial_t u, \nabla q)_K + (g, \nabla q)_K.$$

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$$(\nabla p, \nabla q)_K = -(\partial_t u, \nabla q)_K + (g, \nabla q)_K.$$

Problem

For all $K \in \mathcal{T}_h$, $t > 0$ find $\widetilde{p}_h(t) \in P_1(K)$ such that

$$\begin{aligned} (\nabla \widetilde{p}_h(t), \nabla \widetilde{q}_h)_K &= -(\partial_t u_h(t), \nabla \widetilde{q}_h)_K + (g(t), \nabla \widetilde{q}_h)_K \quad && \forall \widetilde{q}_h \in P_1(K) \\ (\widetilde{p}_h(t), q_h)_K &= (p_h(t), q_h)_K && \forall q_h \in P_0(K), \end{aligned}$$

R. Stenberg *Postprocessing schemes for some mixed finite elements*. RAIRO Model. Math. Anal. Numer. 1991

Y. Chen *Global superconvergence for a mixed finite element method for the wave equation*. Systems Sci. Math. Sci. 1999

Post-Processing

Theorem

For (p, u) sufficiently smooth, we have

$$\|p(t) - \widetilde{p}_h(t)\|_{0,\Omega} \leq C(p, u)h^2$$

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We split the error

$$\begin{aligned}\|p - \widetilde{p}_h\|_{0,K} &\leq \|(p - \pi_1 p)\|_{0,K} + \|\pi_0(\pi_1 p - \widetilde{p}_h)\|_{0,K} + \|(\mathrm{Id} - \pi_0)(\pi_1 p - \widetilde{p}_h)\|_{0,K}. \\ &\leq \|(p - \pi_1 p)\|_{0,K} + \|\pi_0 p - p_h\|_{0,K} + h_K \|\nabla(\pi_1 p - \widetilde{p}_h)\|_{0,K}.\end{aligned}$$

Post-Processing

Theorem

For (p,u) sufficiently smooth, we have

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&\leq \|(p - \pi_1 p)\|_{0,K} + \|\pi_0 p - p_h\|_{0,K} + h_K\|\nabla(\pi_1 p - \widetilde{p}_h)\|_{0,K}.
\end{aligned}$$

We compute

$$\begin{aligned}
(\nabla(\pi_k p - \widetilde{p}_h), \nabla\widetilde{q}_h)_K &= (\nabla(\pi_k p - p), \nabla\widetilde{q}_h)_K + (\nabla(p - \widetilde{p}_h), \nabla\widetilde{q}_h)_K \\
&= (\nabla(\pi_k p - p), \nabla\widetilde{q}_h)_K - (\partial_t(u - u_h), \nabla\widetilde{q}_h)_K \\
&\leq (\|\nabla(\pi_k p - p)\|_{0,K} + \|\partial_t(u - u_h)\|_{0,K})\, \|\nabla\widetilde{q}_h\|_{0,K}.
\end{aligned}$$

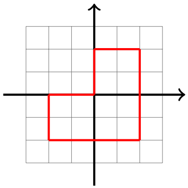
Remarks

- Extension to the a fully discrete scheme
- Generalisation to $Q_h = P_k$ and $V_h = BDM_{k+1}$
- Non-constant coefficients

$$
\begin{aligned}
a\partial_t p + \operatorname{div} u &= f \\
b\partial_t u + \nabla p &= g
\end{aligned}
$$

Numerical Results

Let $\Omega = (-1,1)^2 \setminus [(0,1)\times(-1,0)]$ as visualised below



and take

$$p(x,y,t) = \sin(\pi x)\sin(\pi y)\left(\sin\left(\pi t\sqrt{2}\right) + \cos\left(\pi t\sqrt{2}\right)\right)$$

$$u(x,y,t) = -\frac{\sqrt{2}}{2}\left(\sin\left(\pi t\sqrt{2}\right) - \cos\left(\pi t\sqrt{2}\right)\right)\begin{pmatrix}\cos(\pi x)\sin(\pi y)\\ \sin(\pi x)\cos(\pi y)\end{pmatrix}$$

Choosing a set of basis functions $\{\varphi_i\}_i \subseteq \mathrm{P}_0(\mathcal{T}_h)$ and $\{\Phi_i\}_i \subseteq \mathrm{BDM}_1(\mathcal{T}_h)$ we can rewrite VFD in the form

$$
\begin{aligned}
M\bar{\partial}_\tau \mathbf{u}_h^{n+\frac{1}{2}} + B\mathbf{p}_h^{n+\frac{1}{2}} &= 0 \\
D\bar{\partial}_\tau \mathbf{p}_h^{n+\frac{1}{2}} - B^\top \mathbf{u}_h^{n+\frac{1}{2}} &= 0
\end{aligned}
$$

where $M_{ij} = (\Phi_i, \Phi_j)$, $D_{ij} = (\varphi_i, \varphi_j)$, $B_{ij} = (\operatorname{div}(\Phi_i), \varphi_j)$ and $(\mathbf{p}_h^n, \mathbf{u}_h^n)$ are vectors corresponding to the coefficients of the basis functions.

Numerical Results

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where $M_{ij} = (\Phi_i, \Phi_j)$, $D_{ij} = (\varphi_i, \varphi_j)$, $B_{ij} = (\mathrm{div}\,(\Phi_i), \varphi_j)$ and $(\mathbf{p}_h^n, \mathbf{u}_h^n)$ are vectors corresponding to the coefficients of the basis functions.

$$\begin{pmatrix} \frac{1}{\Delta t} D & -\frac{1}{2} B^\top \\ \frac{1}{2} B & \frac{1}{\Delta t} M \end{pmatrix} \begin{pmatrix} \mathbf{p}_h^{n+1} \\ \mathbf{u}_h^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t} D & \frac{1}{2} B^\top \\ -\frac{1}{2} B & \frac{1}{\Delta t} M \end{pmatrix} \begin{pmatrix} \mathbf{p}_h^{n} \\ \mathbf{u}_h^{n} \end{pmatrix}$$

Numerical Results

Convergence with respect to h for a fixed time step $\Delta t = 0.001$ and $n\Delta t = T = 1$.

h	$\|\pi_{L^2} u^n - u_h^n\|_{L^2}$	rate	$\|\pi_{L^2} p^n - p_h^n\|_{L^2}$	rate
0.5	4.481e-01	-	3.003e-01	-
0.25	1.654e-01	1.4379	6.533e-02	2.2009
0.125	4.968e-02	1.7353	1.884e-02	1.7941
0.0625	1.293e-02	1.9418	4.883e-03	1.9478
0.0313	3.276e-03	1.9808	1.191e-03	2.0355

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Convergence with respect to Δt for a fixed $h = 2^{-9}$ and $n\Delta t = T = 1$.

Δt	$\|\pi_{L^2} u^n - u_h^n\|_{L^2}$	rate	$\|\pi_{L^2} p^n - p_h^n\|_{L^2}$	rate
0.1	3.998e-02	-	2.101e-02	-
0.05	1.064e-02	1.9092	5.916e-03	1.8287
0.025	2.732e-03	1.9618	1.542e-03	1.9399
0.0125	6.880e-04	1.9896	3.901e-04	1.9828
0.00625	1.690e-04	2.0258	9.626e-05	2.0189

Numerical Results

Convergence of PP w.r.t. h for a fixed time step $\Delta t = 0.001$ and $n\Delta t = T = 1$.

h	$\|\|p^n - \tilde{p}_h^n\|\|_{L^2}$	rate	$\|\|p^n - p_h^n\|\|_{L^2}$	rate
0.5	4.630e-01	-	6.770e-01	-
0.25	1.092e-01	2.0844	2.919e-01	1.2135
0.125	2.945e-02	1.8904	1.511e-01	0.9506
0.0625	7.467e-03	1.9795	7.595e-02	0.9920
0.0313	1.840e-03	2.0213	3.802e-02	0.9983
0.0156	4.626e-04	1.9915	1.902e-02	0.9995

Numerical Results

Convergence of PP w.r.t. h for a fixed time step $\Delta t = 0.001$ and $n\Delta t = T = 1$.

h	$\|\|p^n - \tilde{p}_h^n\|\|_{L^2}$	rate	$\|\|p^n - p_h^n\|\|_{L^2}$	rate
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0.0625	7.467e-03	1.9795	7.595e-02	0.9920
0.0313	1.840e-03	2.0213	3.802e-02	0.9983
0.0156	4.626e-04	1.9915	1.902e-02	0.9995

Convergence of PP w.r.t. h for a fixed time step $\Delta t = 0.001$ and $n\Delta t = T = 1$.

Δt	$\|\|p^n - \tilde{p}_h^n\|\|_{L^2}$	rate
0.1	2.101e-02	-
0.05	5.910e-03	1.8297
0.025	1.536e-03	1.9438
0.0125	3.847e-04	1.9975
0.00625	9.175e-05	2.0680

The End/ Acknowledgement

Thank you for your attention

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